Nonstochastic Control

Controlling Dynamics Online

Max Simchowitz (CMU) & Elad Hazan (Princeton)

Motivation: ML as Improper Learning

The World is Full of Dynamical Systems

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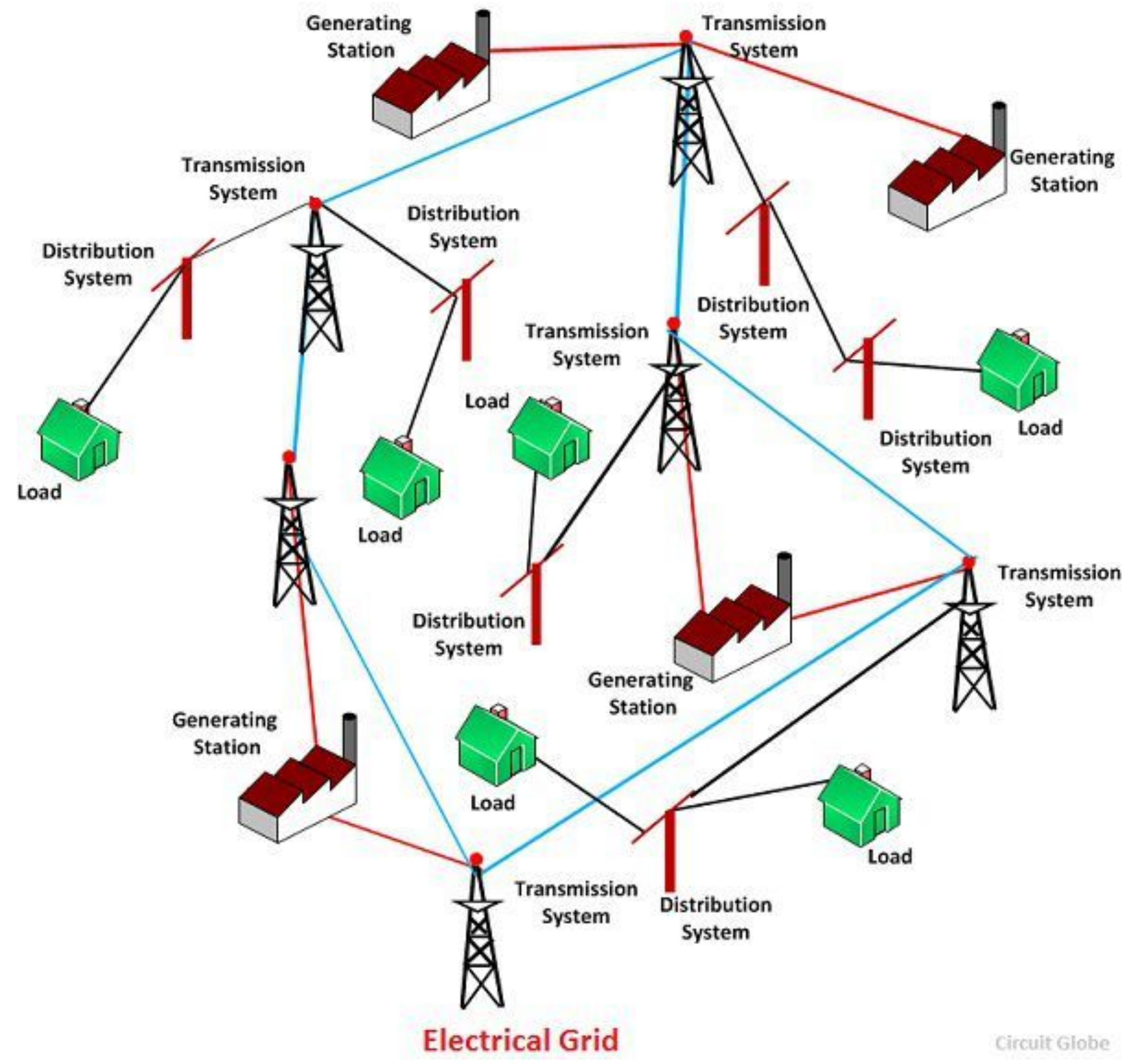
robotics

The World is Full of Dynamical Systems

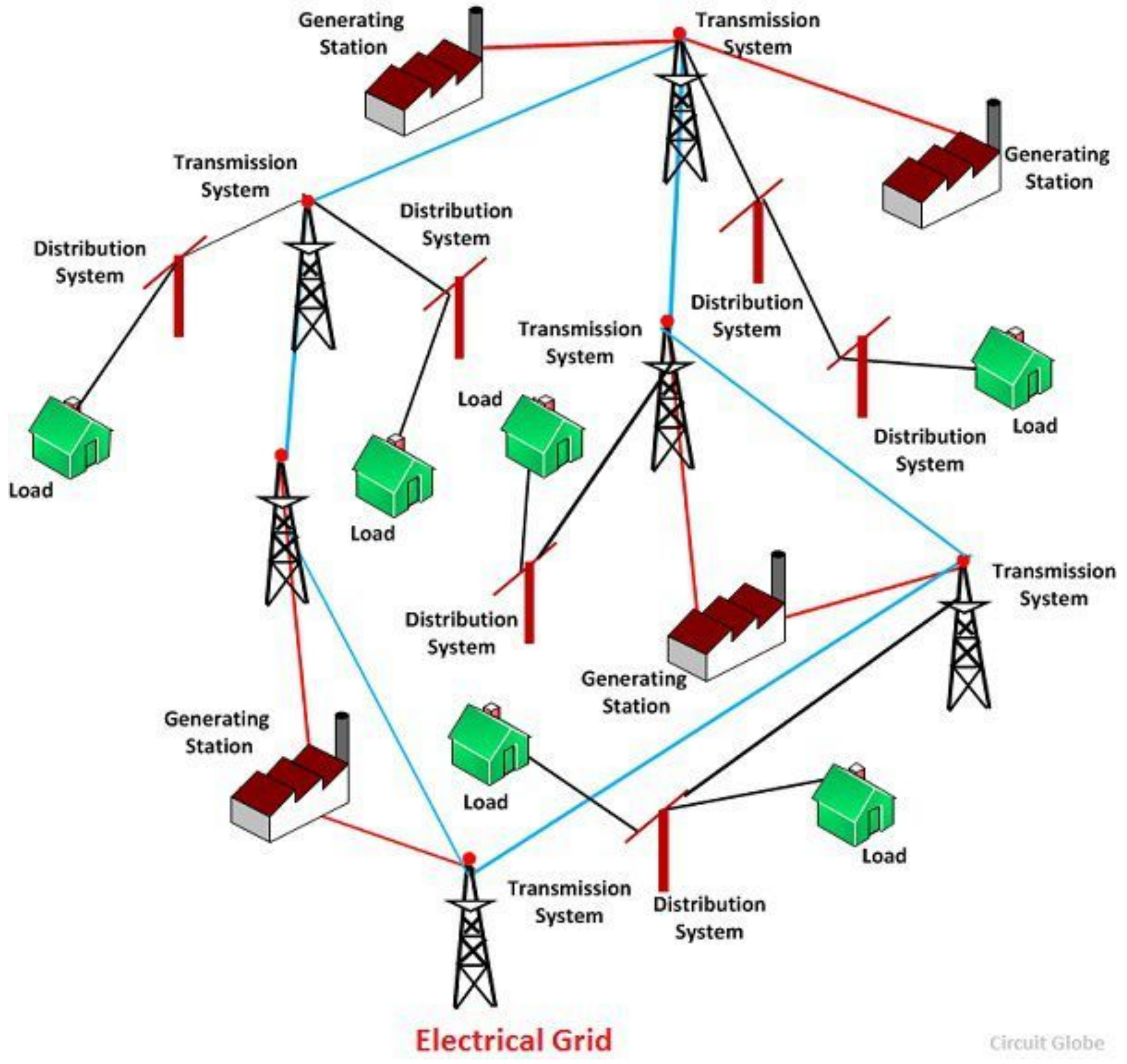
robotics



power grid

The World is Full of Dynamical Systems

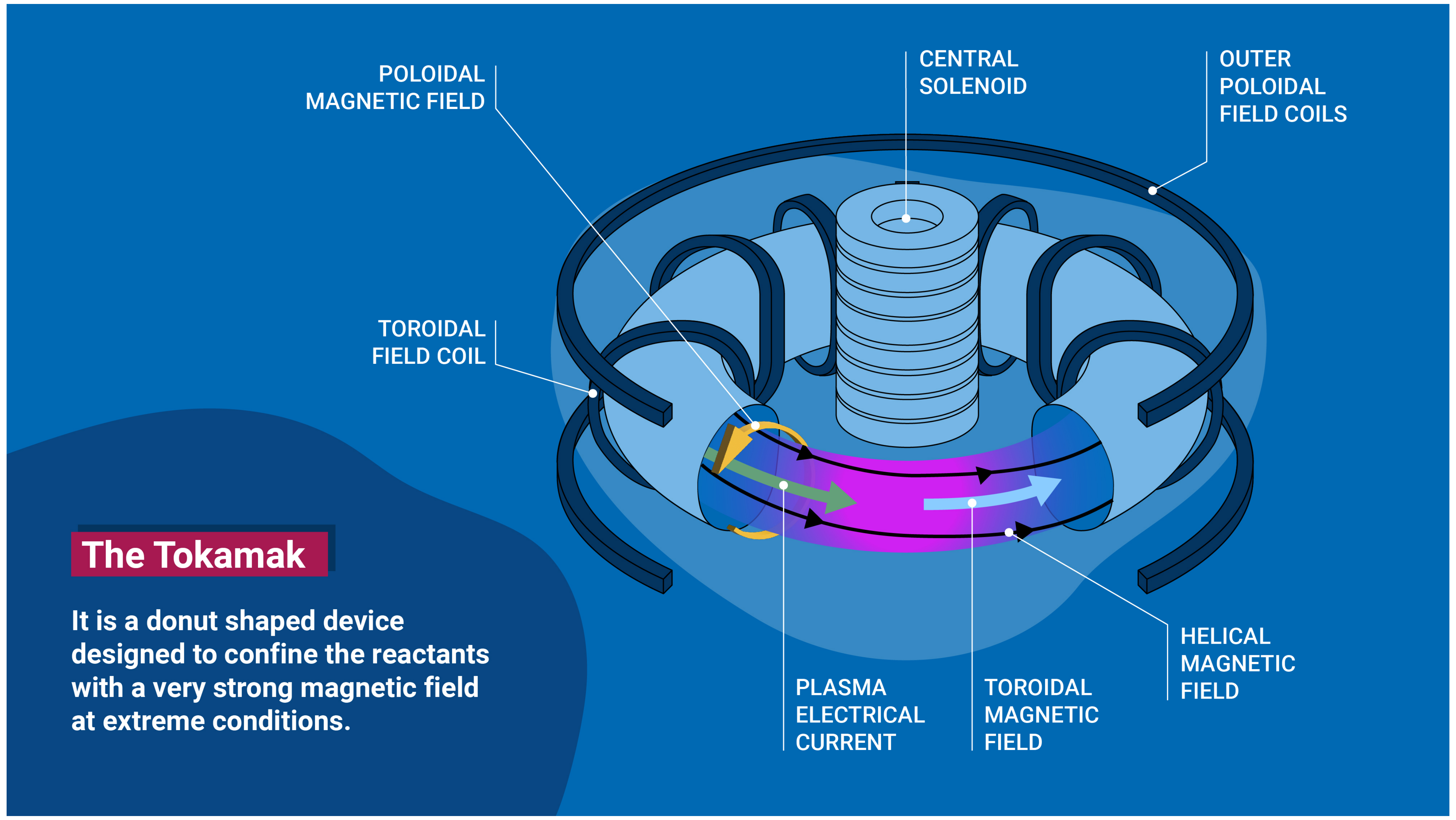
robotics



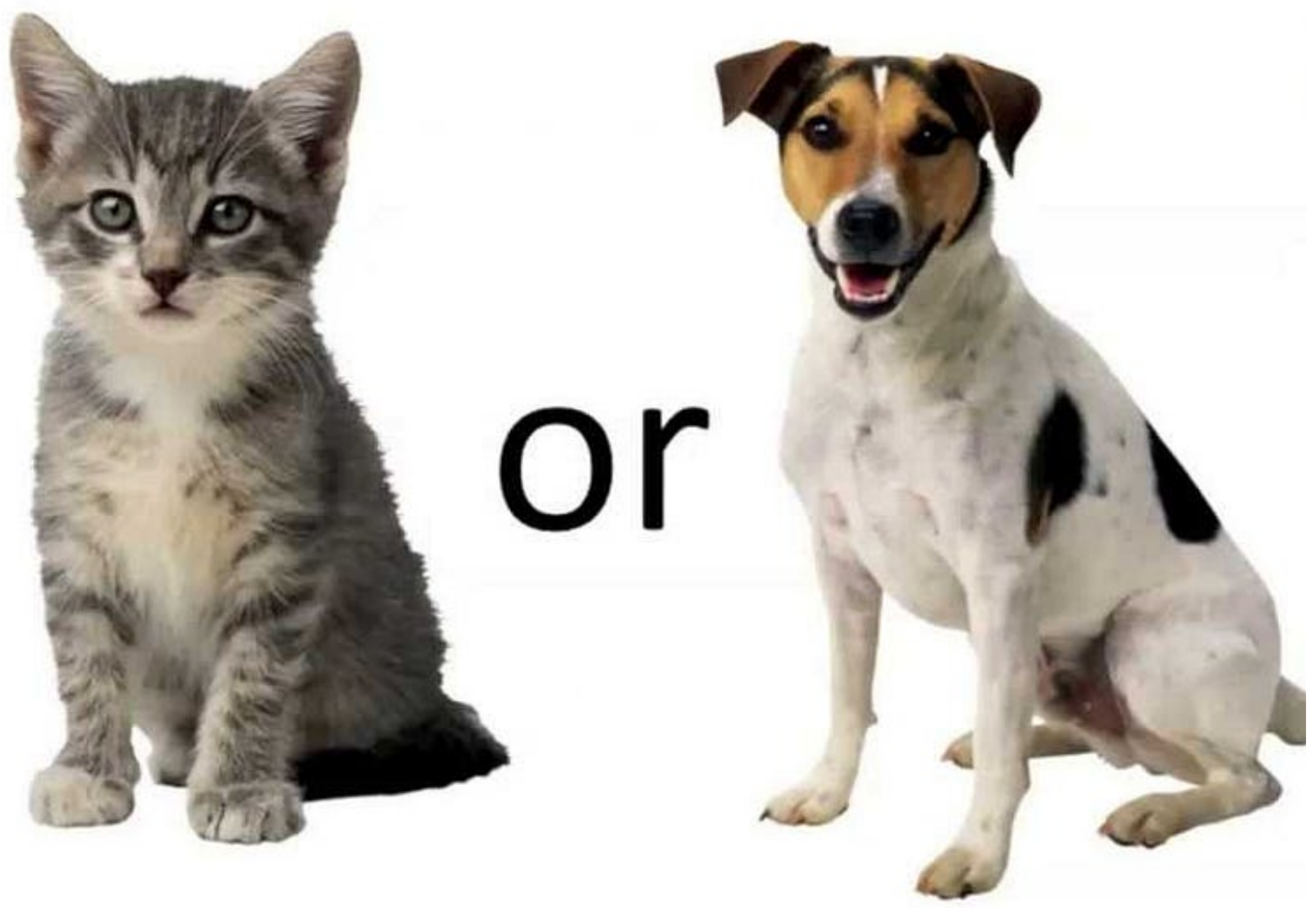
power grid

chemical plants

What about dynamics that are hard to model?



The golden rule of modern machine learning



"If computer vision researchers spent all their time searching for the **correct definition of a "cat"** in 2015, they would have made zero progress"
— **Terry Suh**

***this perspective comes with numerous drawbacks, e.g. robustness**

Applying the golden rule to control

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1. **Learning:** quantities which are unknown can be estimated statistically

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2. **Relaxation/"Improperness:"** learn surrogate models which do not share the same functional form as the ground-truth (e.g. neural dynamics)

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1. **Learning:** quantities which are unknown can be estimated statistically

2. **Relaxation/"Improperness:"** learn surrogate models which do not share the same functional form as the ground-truth (e.g. neural dynamics)

3. **Adaptation:** we can adapt our actions to a changing world.

Applying the golden rule to control

This Tutorial: A Mathematical Formalism for Control that combines **learning**, **improperness**, and **adaption**.

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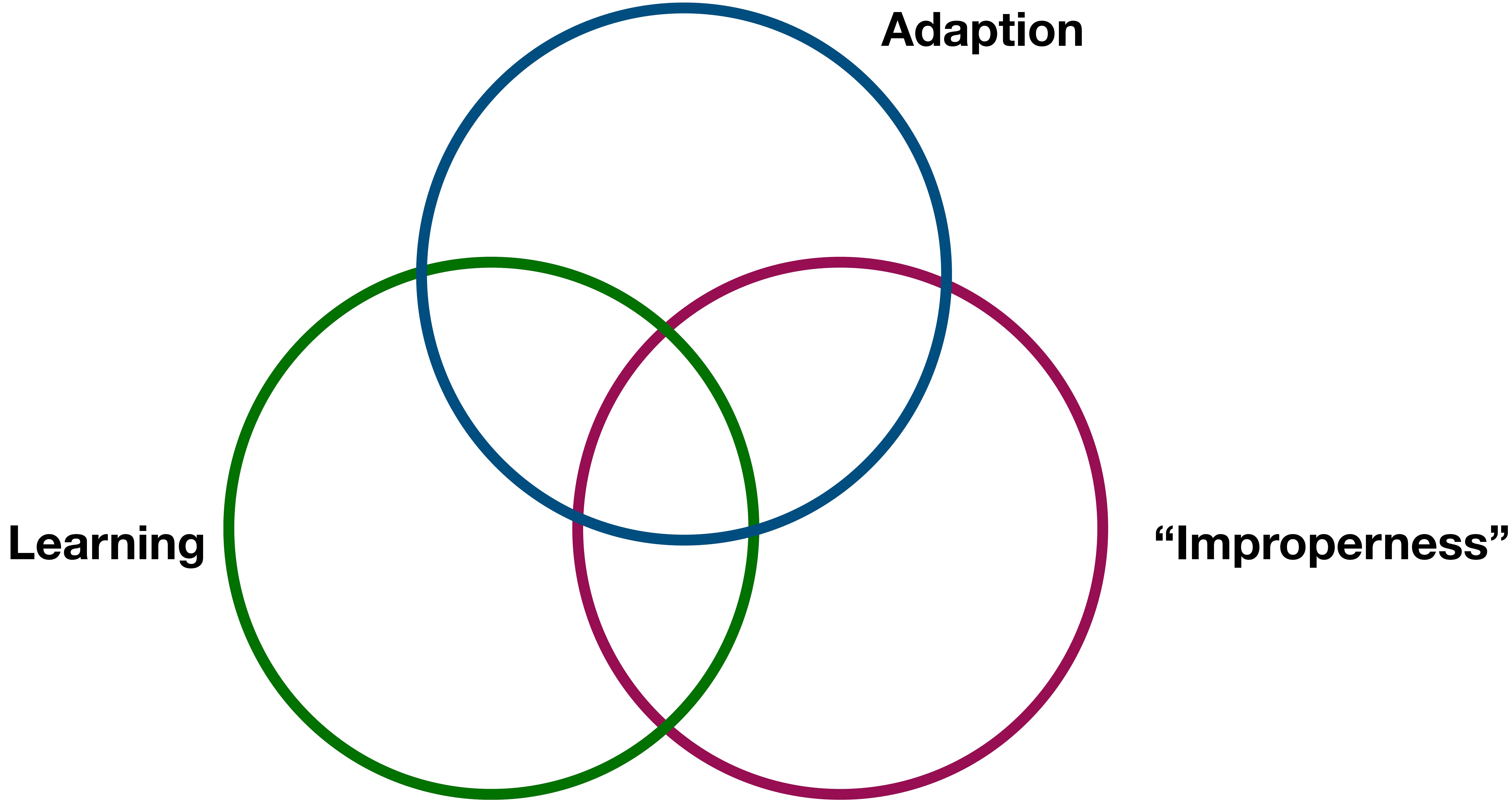

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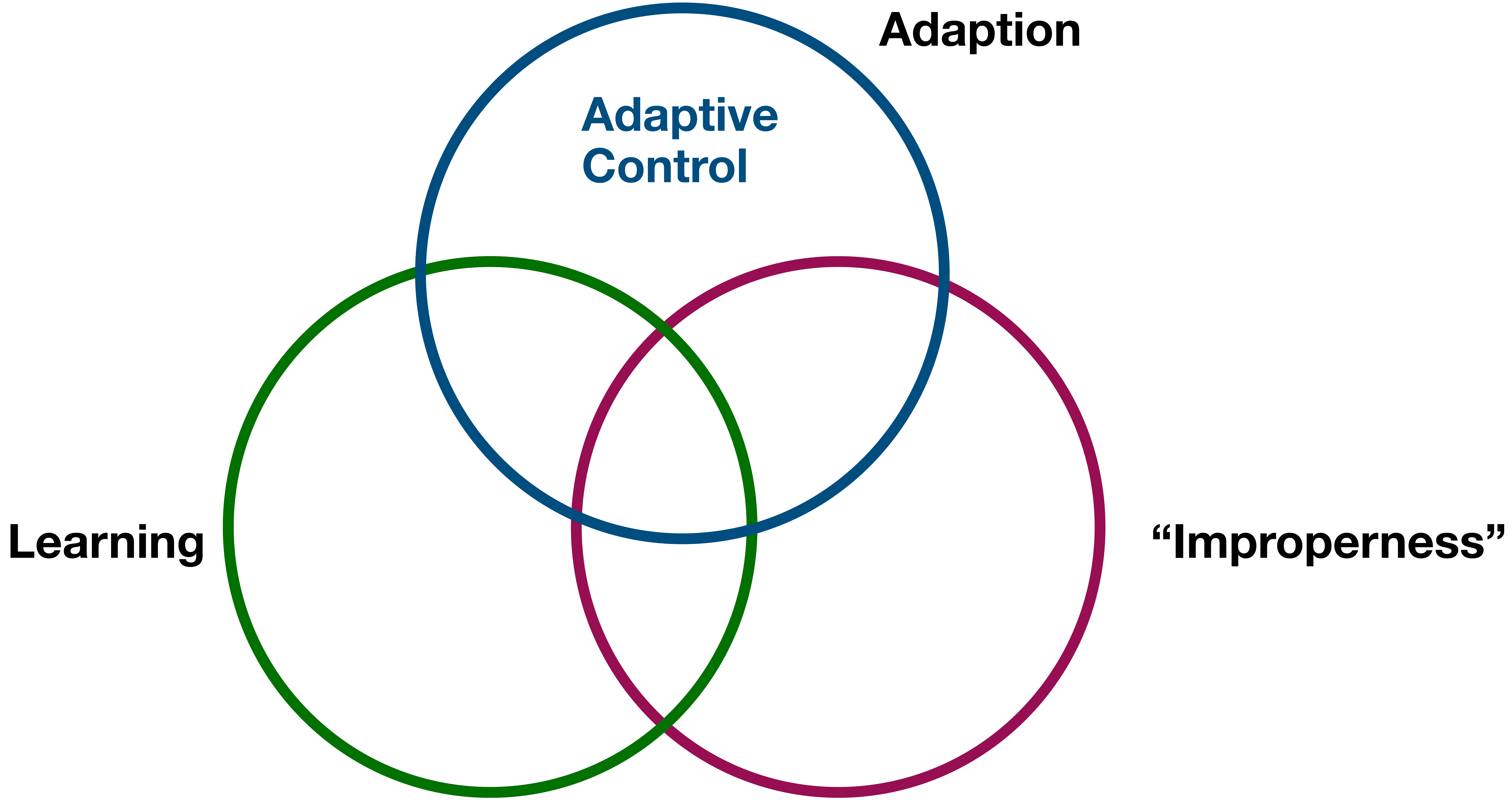

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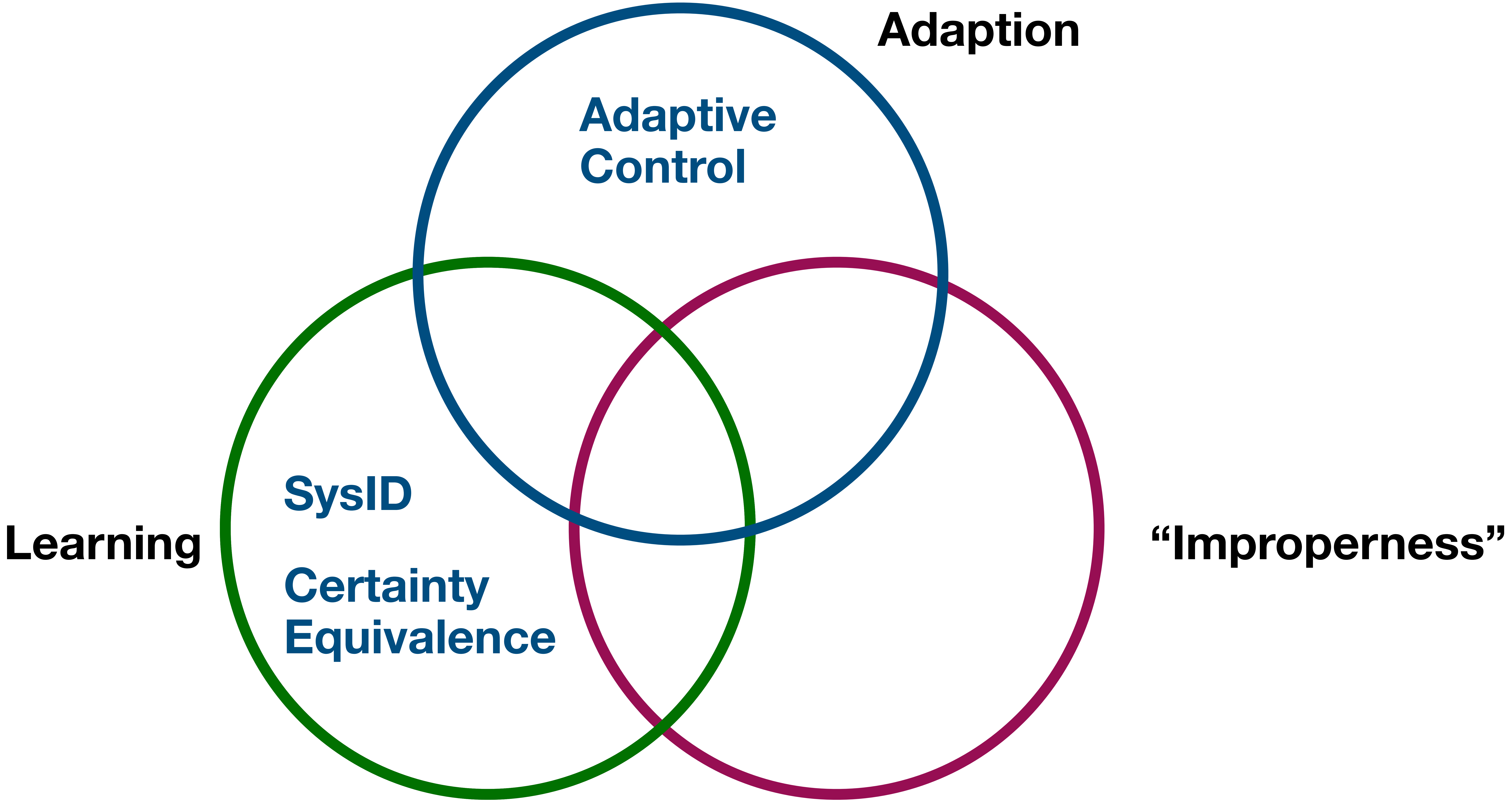

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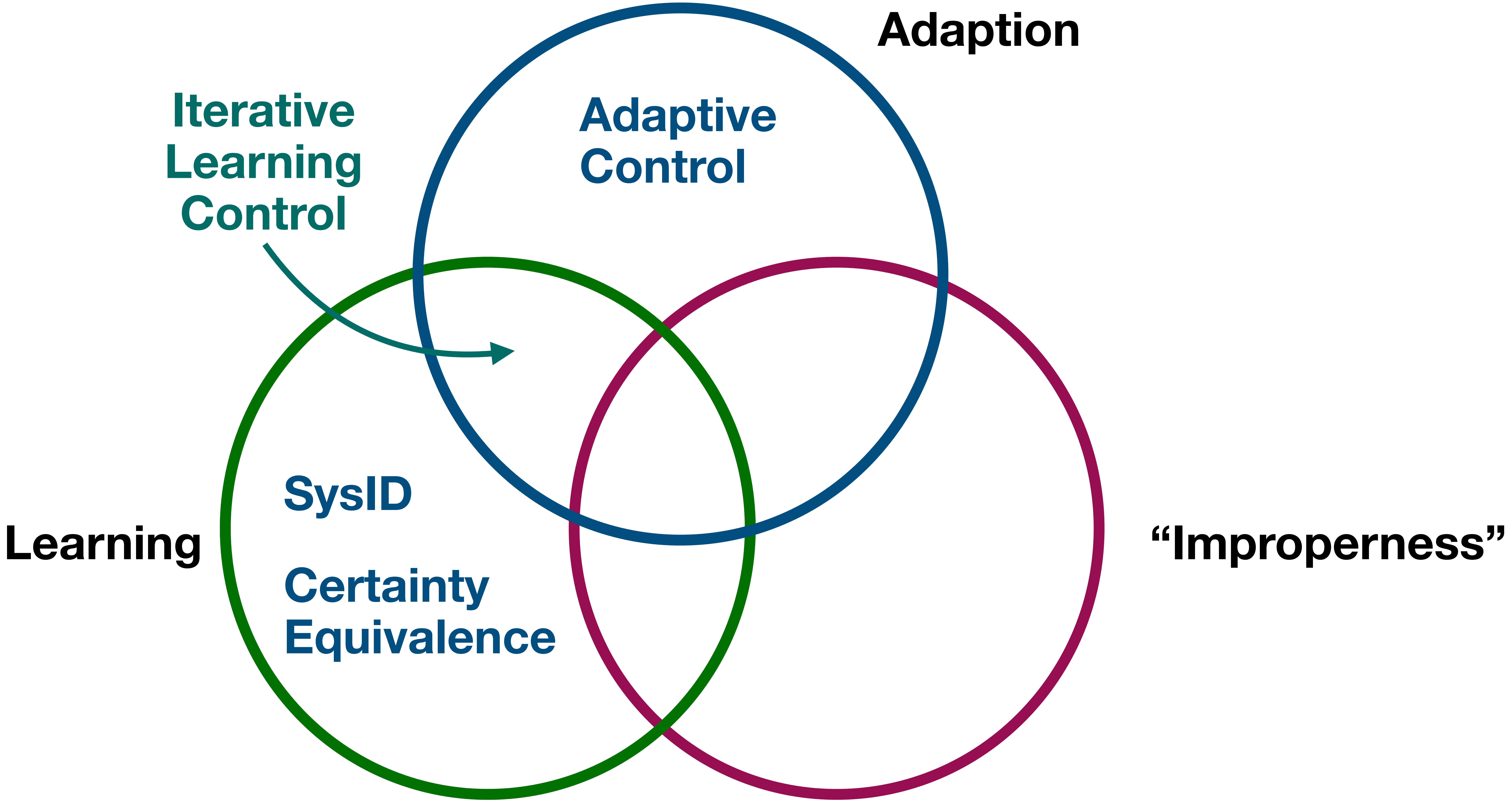

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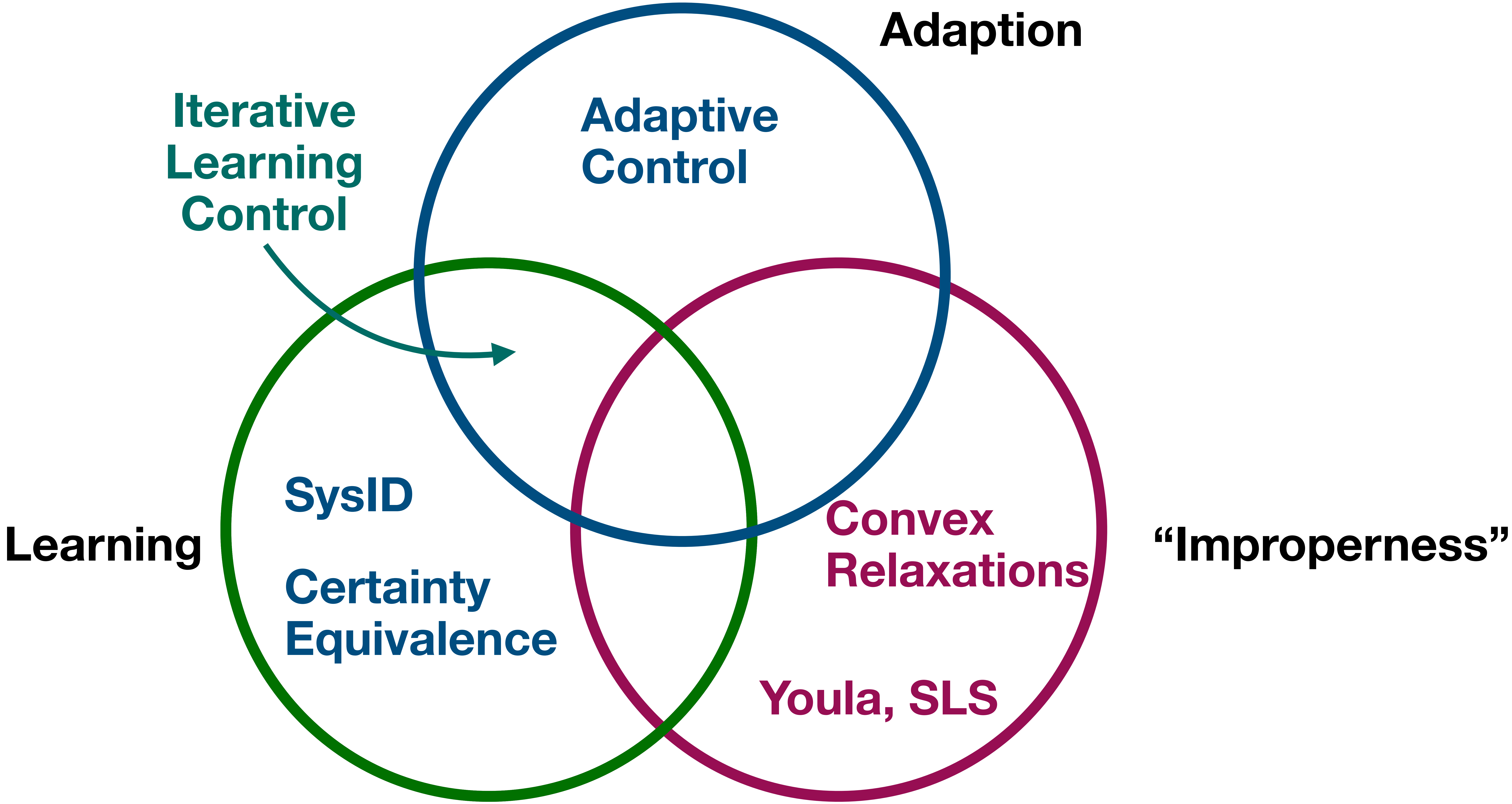
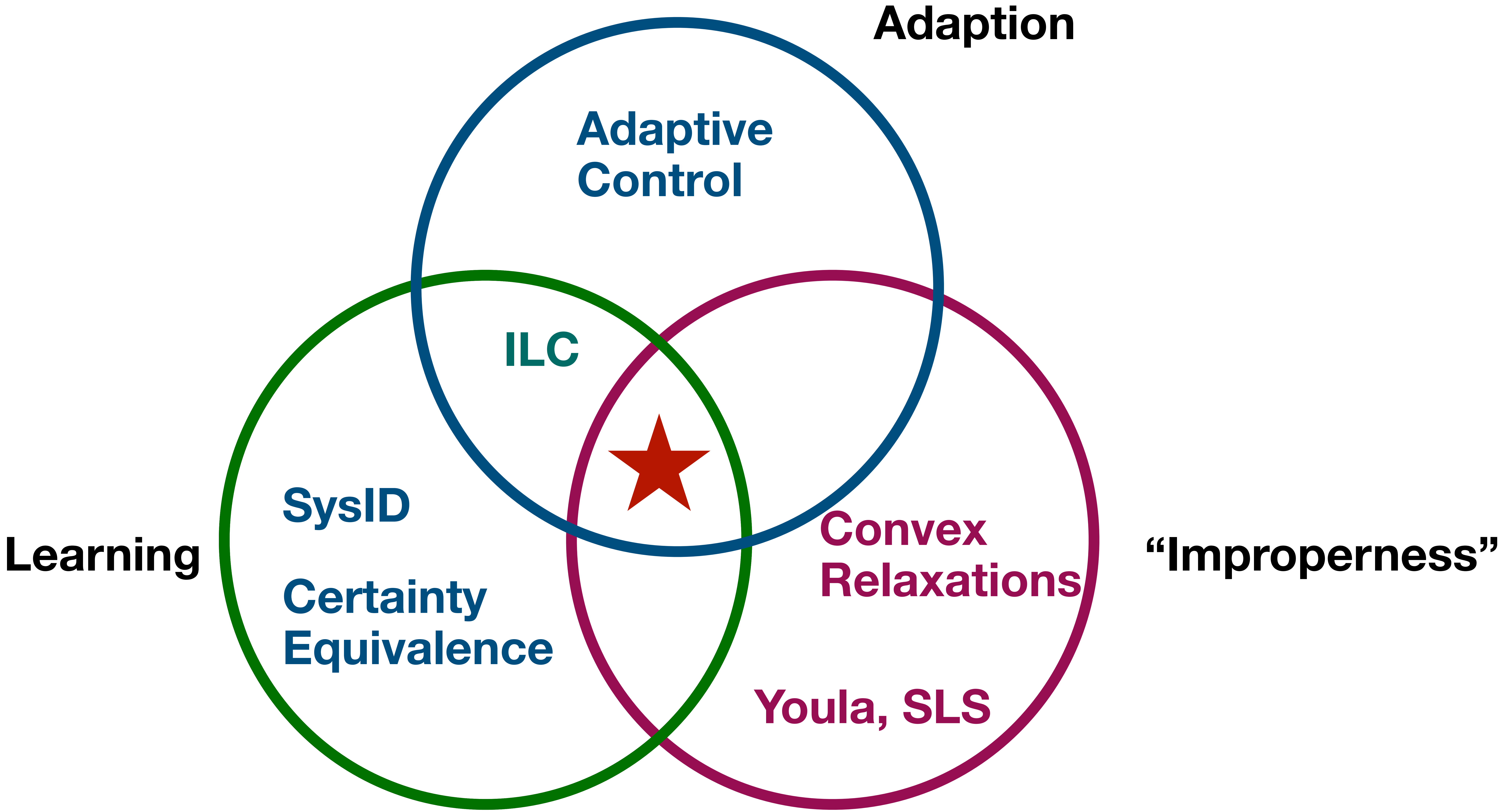


Non-stochastic control at the intersection



Core Concepts:

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3. Combine statistical learning with online optimization

Basics of Classical Control

Background: Dynamical Systems

Recall: A **dynamical system** is

$$\boxed{x_{t+1} = f(x_t, u_t) + w_t}$$

state **control input** **disturbance**

Background: Dynamical Systems

Recall: A **dynamical system** is

$$x_{t+1} = f(x_t, u_t) + w_t$$

$$y_t = g(x_t) + e_t$$

dynamics model

observation model

observation noise

Control As an Interactive Protocol

For each time t,

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Goal: For a given cost $c(\cdot\,,\cdot)$, make $J_T = \sum_{t=1}^{T} c(y_t, u_t)$ as small as possible.

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what does this mean?

Agent's 'Strategy': A Control Policy

If **dynamics** and $W := (w_{1:T}, e_{1:T})$ known beforehand, can directly* optimize

$$\boxed{J_T = \sum_{t=1}^{T} c(y_t, u_t)}$$

might be hard computationally

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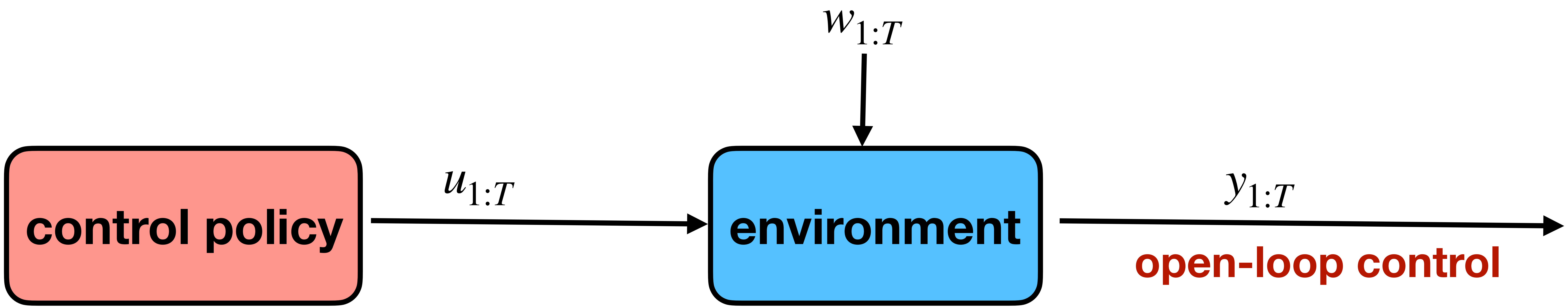

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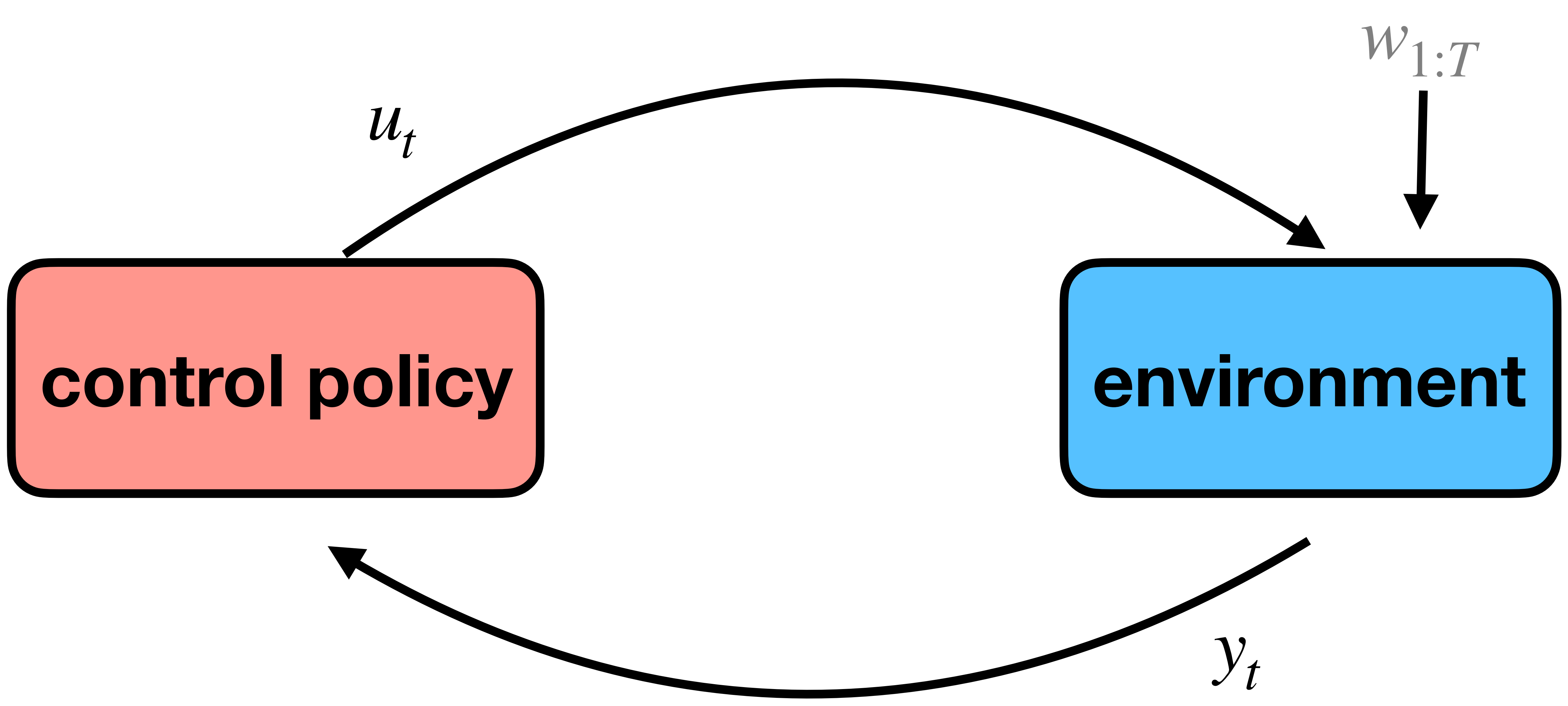

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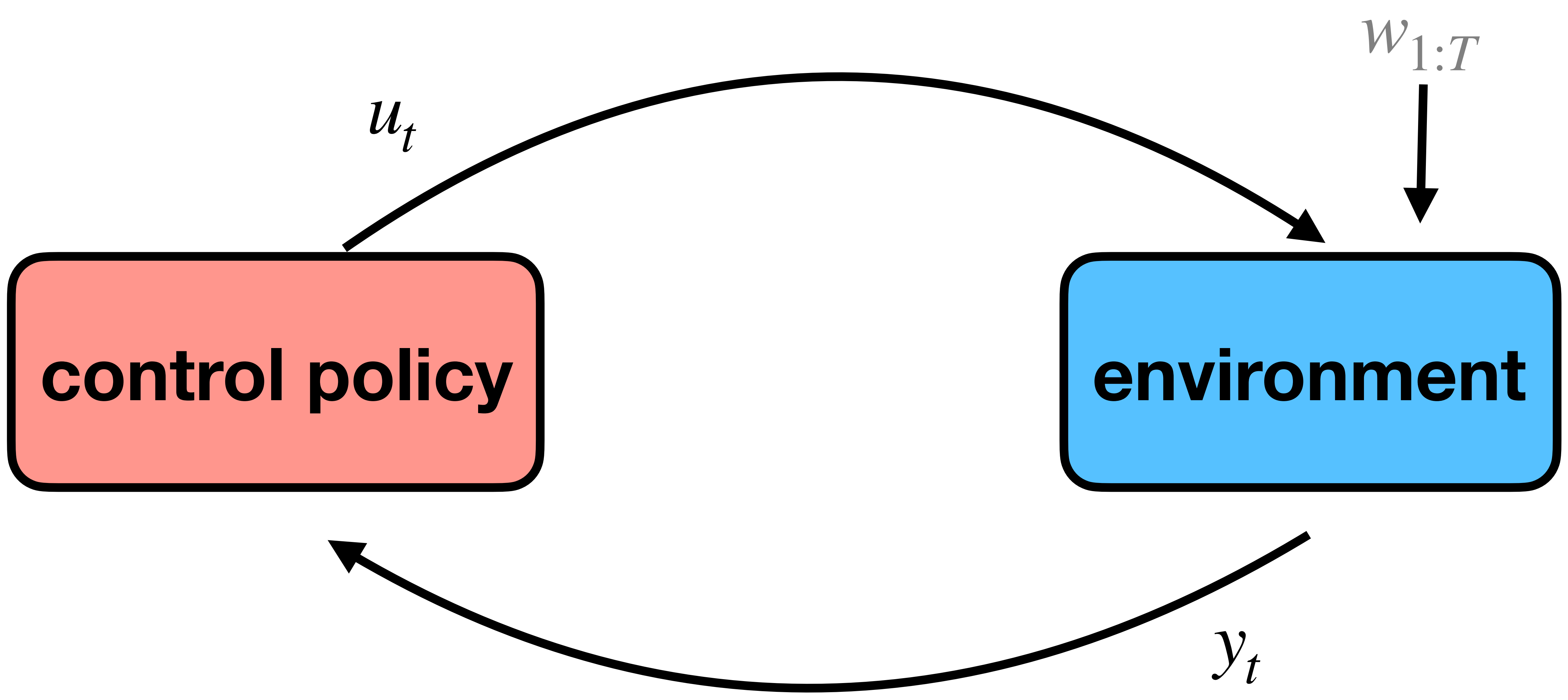

1. **History Dependent:** $\pi : (y_{1:t}, u_{1:t-1}) \to u_t$
2. **State-Based** $\pi : (x_{1:t}, u_{1:t-1}) \to u_t$
3. **State-Feedback** $\pi : x_t \to u_t$

Background: Control Cost

Recall: For a fixed dynamical system, the **control cost** of a policy π is

$$\boxed{J_T(\pi; W) = \sum_{t=1}^{T} c(y_t, u_t)}$$

control-cost

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2. $u_t = \pi(y_{1:t}, u_{1:t-1})$
3. $W = (w_{1:T}, e_{1:T})$

Background: The Optimal Control Problem

The optimal control problem is

$$\min_{\pi} \mathbb{O}[J_T(\pi; W)]$$

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describes the W

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1. **fixed** W **(open-loop planning/trajectory opt.)**
2. **random** $\mathbb{E}_W$ **(stochastic optimal control)**
3. **worst-case** $\sup_{W \in \ldots}$ **(robust control**, e.g. the work of John Doyle**)**

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4. We briefly described **classical noise models** (fixed, random, worst-case).

Basics of Linear Control

Background: Linear Dynamical System

Recall: A **linear dynamical system** is

$$x_{t+1} = Ax_t + Bu_t + w_t$$

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Rationale: Local Taylor Approximation of Nonlinear Dynamics.

Linear Quadratic Optimal Control Problems

Linear Quadratic Optimal Control

$$J_T(\pi; W) = \sum_{t=1}^{T} c(y_t, u_t)$$

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convex quadratic: $Q, R \succeq 0$

Linear Quadratic Optimal Control Problems

Classical Linear Quadratic Optimal Control

Stochastic Control

Robust Control

Linear Quadratic Optimal Control Problems

Classical Linear Quadratic Optimal Control

$$\min_{\pi} \lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{w,e}[J_T(\pi; W)]$$

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The $\mathscr{H}_2$ control problem: w_t, e_t are i.i.d Gaussian (*Kalman, LQG*)

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Robust Control

The $\mathscr{H}_\infty$ control problem:
w_t, e_t are worst case (*Doyle*)

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Classical Linear Quadratic LQ Optimal Control

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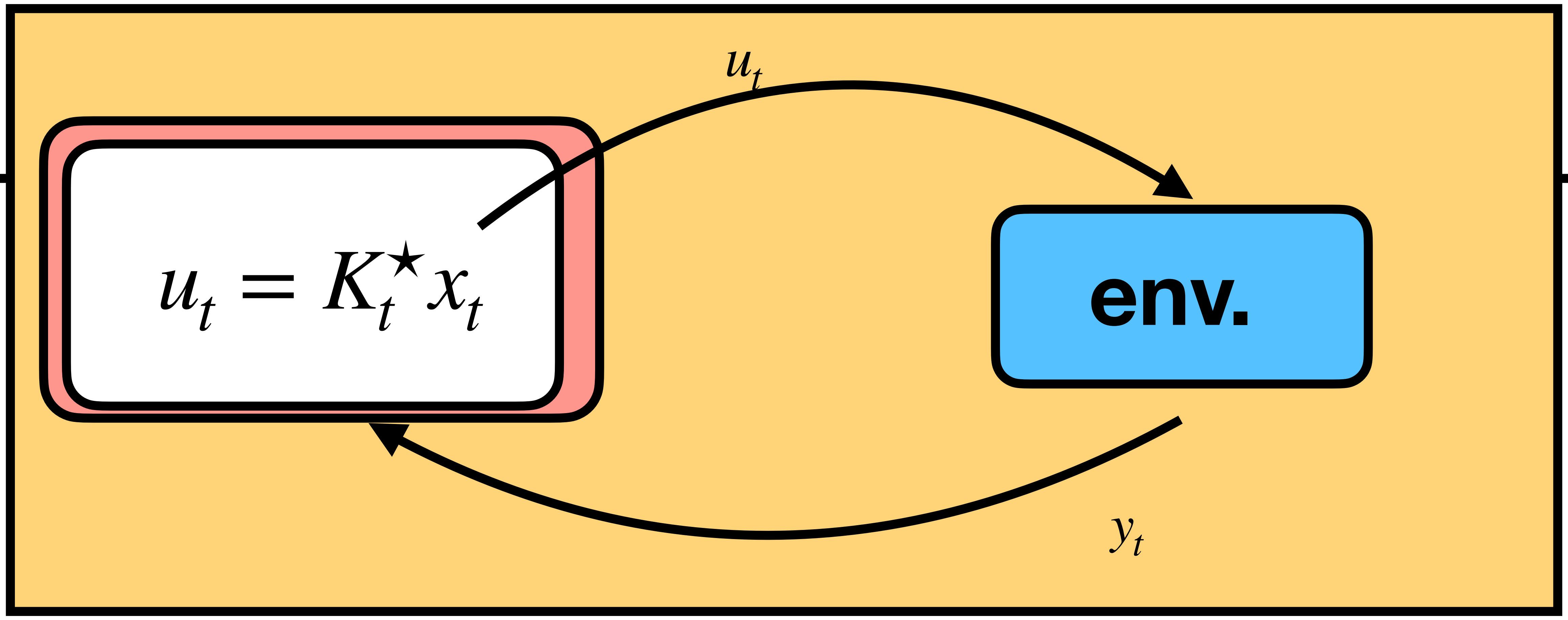
Theorem: If **fully observed** $(y_t \equiv x_t)$, **state-feedback is optimal**



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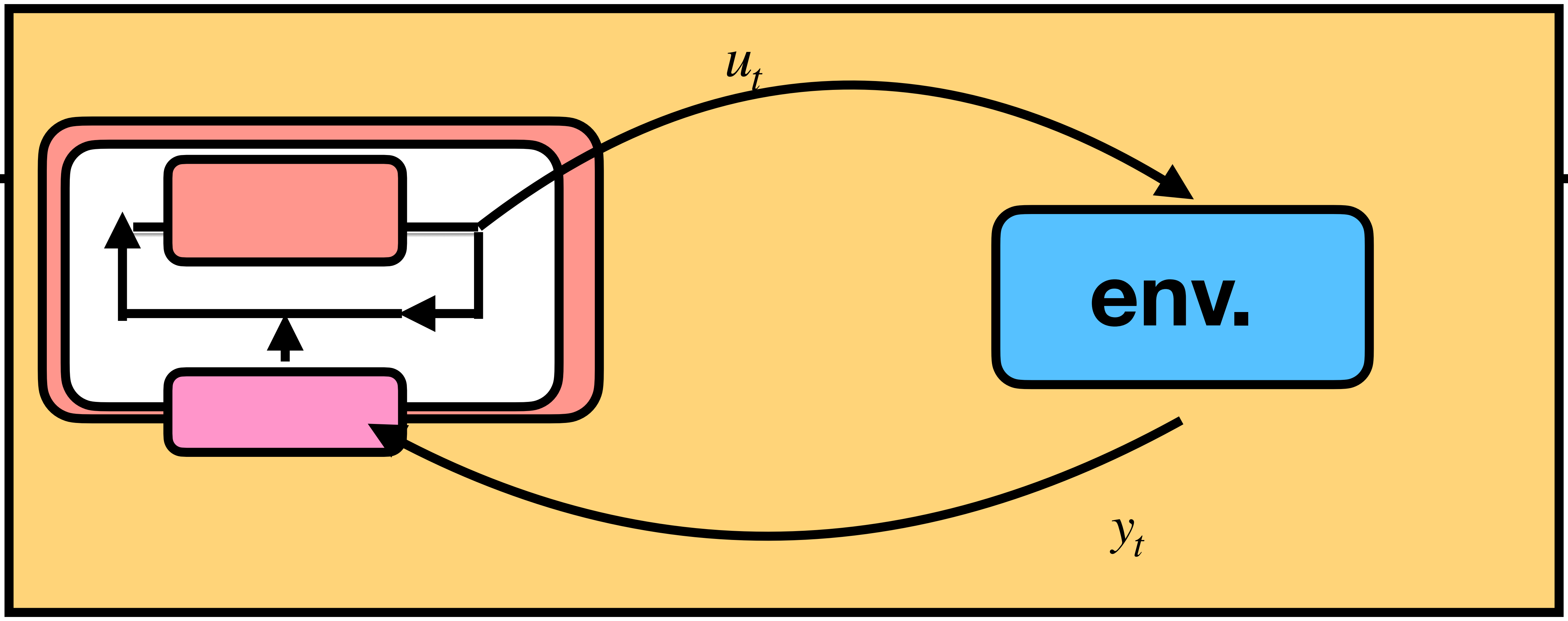
Theorem: For **general LQ control** are **linear dynamic policies** are optimal**:**



$$z_{t+1} = A_\pi z_t + B_\pi y_t$$

$$u_t = C_\pi z_t + D_\pi y_t$$

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Important Takeaway: Linear Quadratic Control Problems admit **easy-to-express** controllers.

Beyond LQ Control

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Challenge 2: Optimizing over **feedback** (static or dynamic) is **non-convex** and can be computationally hard:

This is because, e.g. in full observation $x_t = \sum_s (A + BK)^{t-s}(Bu_s + w_s)$

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Challenges: Direct optimization over feedback controllers can be hard, and exact optimal control laws can be hard to express.

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Powerful Observation: Youla-Kućera ’76, Zames ’81 (IO), Anderson et al. ’19 (SLS)

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4. We hinted at **convex relaxations** as a tool for efficient optimization.

The Non-Stochastic Control Problem

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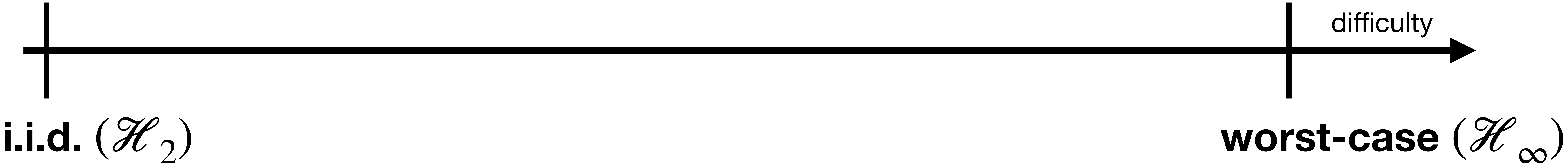

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"What π would we pick if we knew noise W in hindsight"

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But of course: impossible, and leads to open loop control

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We will allow adversarial noise, but introduce **regret** to measure performance

Solution Concept: Regret

“i’ve had a few”

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counterfactual cost under policy $\pi \in \Pi$

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best-in-hindsight

(with full knowledge of disturbances)

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"competing with Π**"**

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we can embed a **prediction problem** where comparator has zero cost (perfect knowledge), but learner has $\Omega(T)$ cost.

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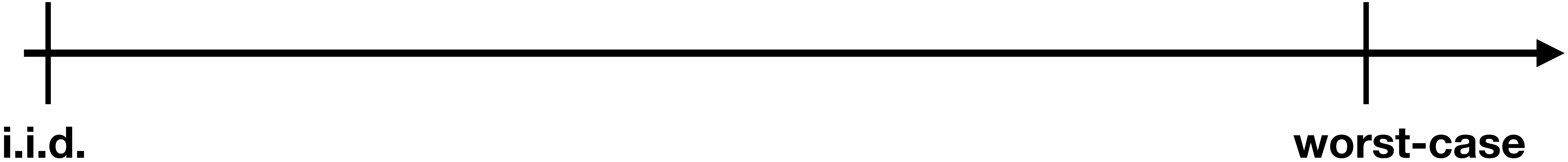
Key Idea: Optimizing over linear policies can **efficient, even when optimal control is not.**

Compared to What? For linear dynamics.

What class of comparator policies $\pi \in \Pi$**?**



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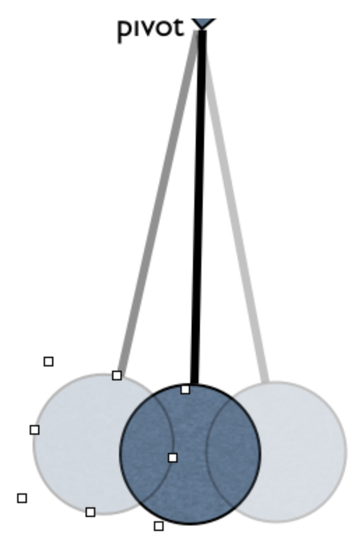

Pendulum

Nonstochastic Control As an Interactive Protocol

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Nonstochastic Control As an Interactive Protocol

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Goal: make $\mathrm{Reg}_T(\mathbb{A}; \Pi) = J_T(\mathbb{A}; W) - \min_{\pi \in \Pi} J_T(\pi; W) = o(T)$.

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Nonstochastic Control As an Interactive Protocol

For each time t,

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defined with changing costs

Linear Nonstochastic Control: Interactive Protocol

For each time t,

1. **Nature picks noise (w_t, e_t) and a cost c_t**
2. **Dynamics reveal $y_t = Cx_t + e_t$**
3. **Control agent picks u_t**
4. **Dynamics evolve $x_{t+1} = Ax_t + Bu_t + w_t$, suffer $c_t(y_t, u_t)$**

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2. We define **regret** to a **restricted comparator class** as a performance yardstick when noise is possibly **adversarial**

3. We formulated the **non-stochastic control** protocol, including changing costs.

Roadmap: Core Challenges

Goal: make $\mathrm{Reg}_T(\mathbb{A};\Pi) = J_T(\mathbb{A};W) - \min_{\pi\in\Pi} J_T(\pi;W)$ **small.**

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1. **GPC**: Fully Observed, Known-Dynamics

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(can be relaxed)

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The Gradient Perturbation Controller (GPC)

Roadmap

1. GPC: Fully Observed, Known-Dynamics

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Warmup: Known System + Stable Dynamics

1. **Fully Observed:** $y_t \equiv x_t$
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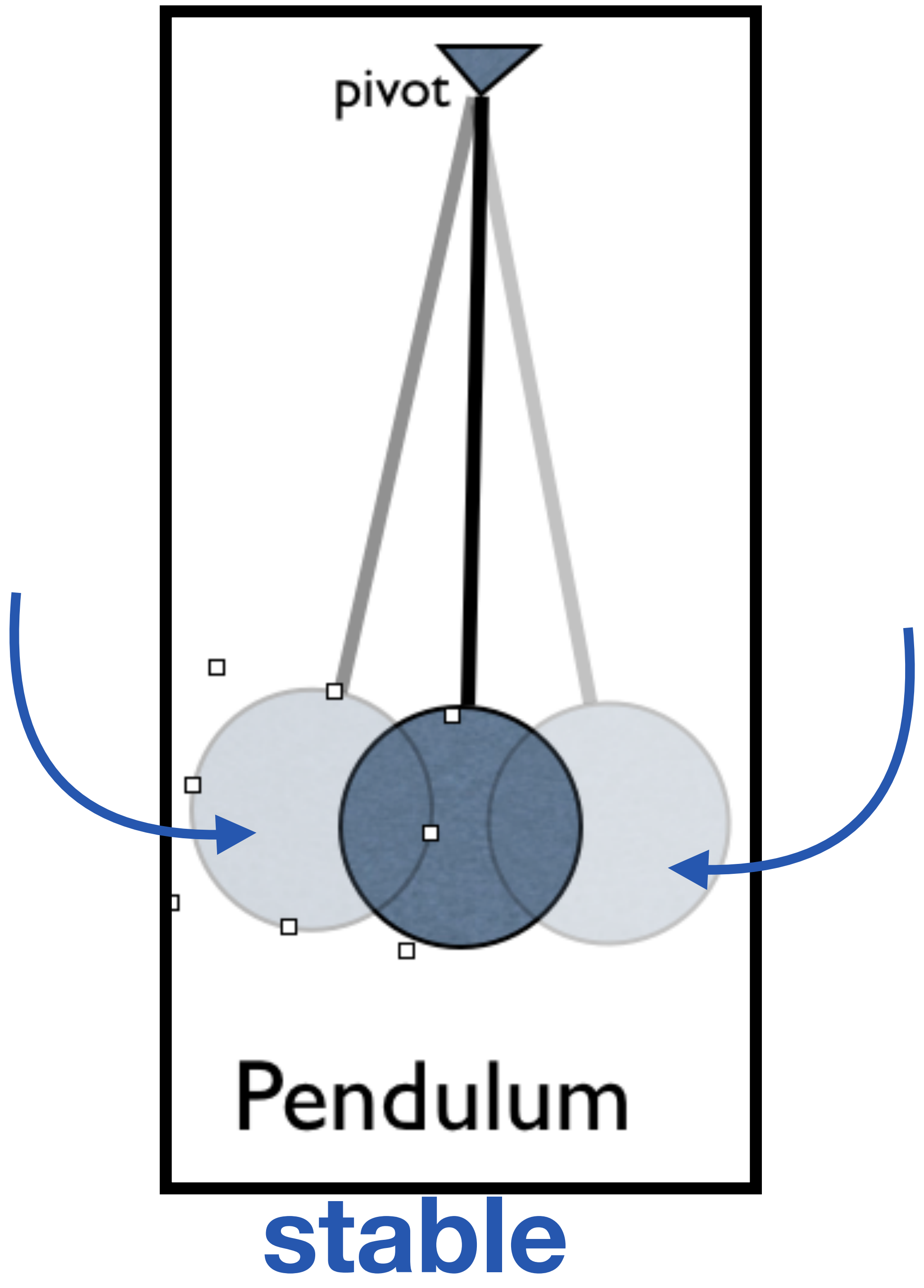

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The Gradient Perturbation Controller

For $t = 1,2,\ldots$

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2. $\boxed{M_{t+1} \leftarrow M_t - \eta_t \nabla \tilde{F}_t(M_t)}$ where $\tilde{F}_t$ is convex **(online gradient descent)**

Goal: Known System + Stable Dynamics

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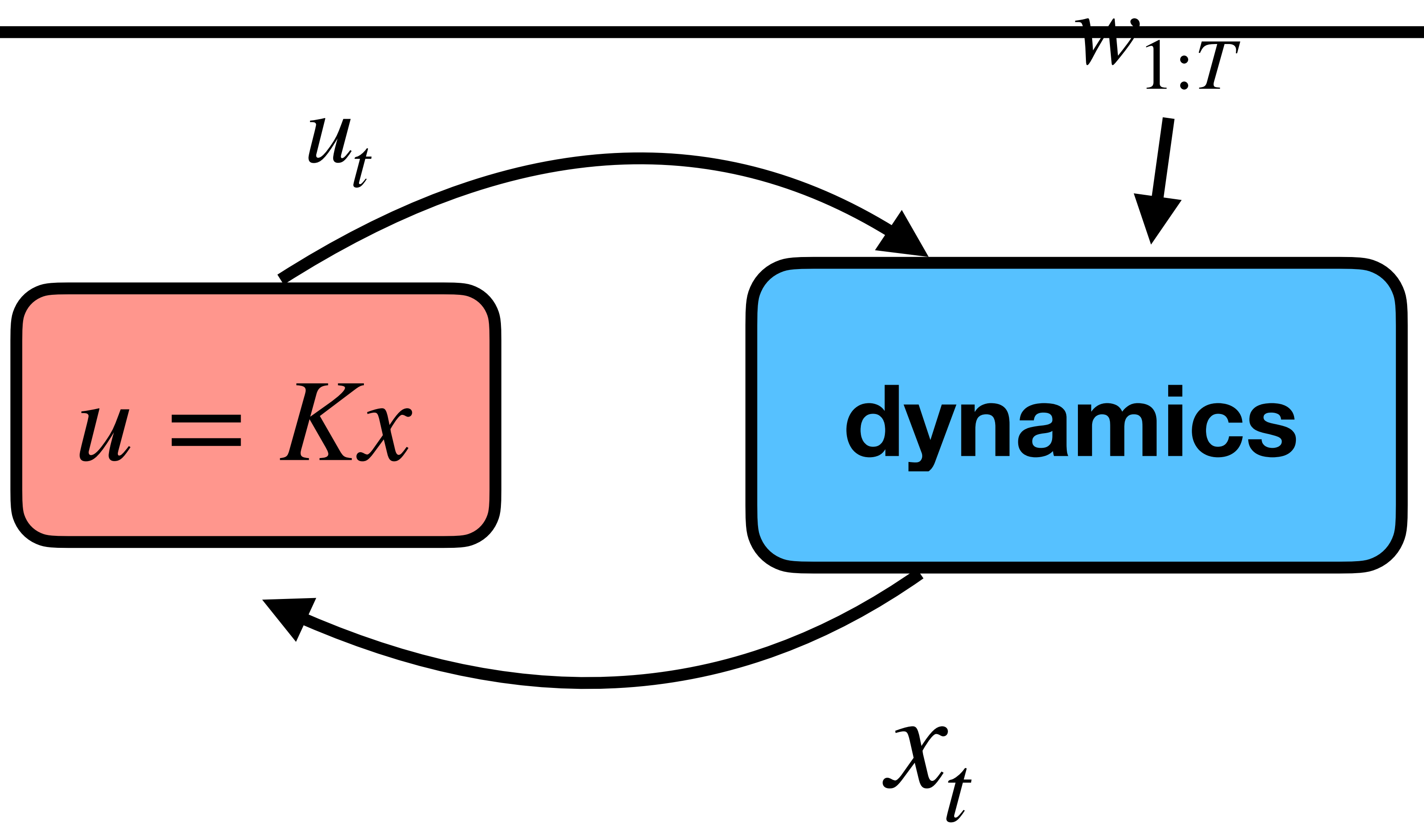
$$\Pi_{\text{feedback}} := \{\pi(x) = Kx : A + BK \text{ is } (C, \rho) \text{ stable})$$



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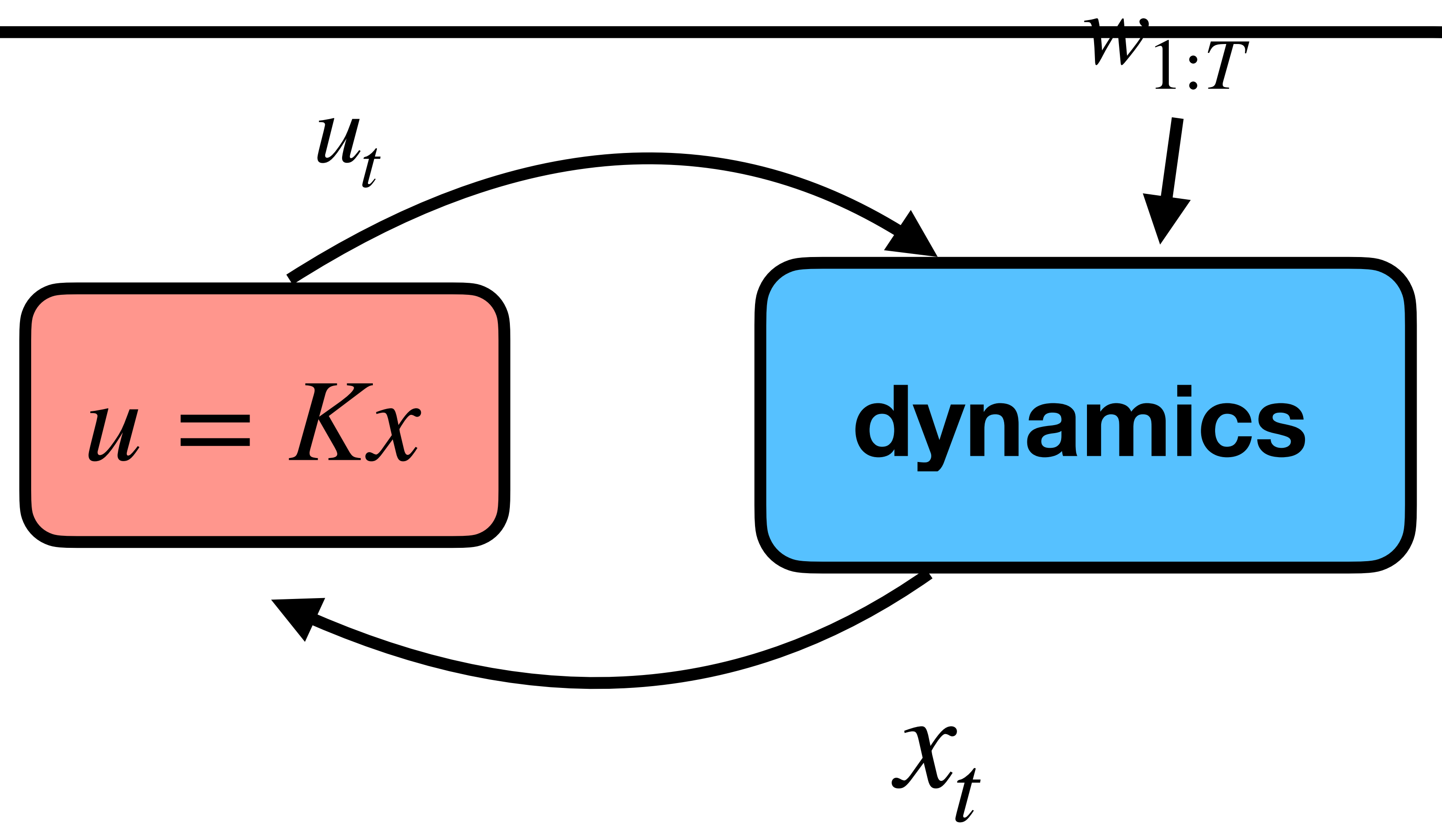

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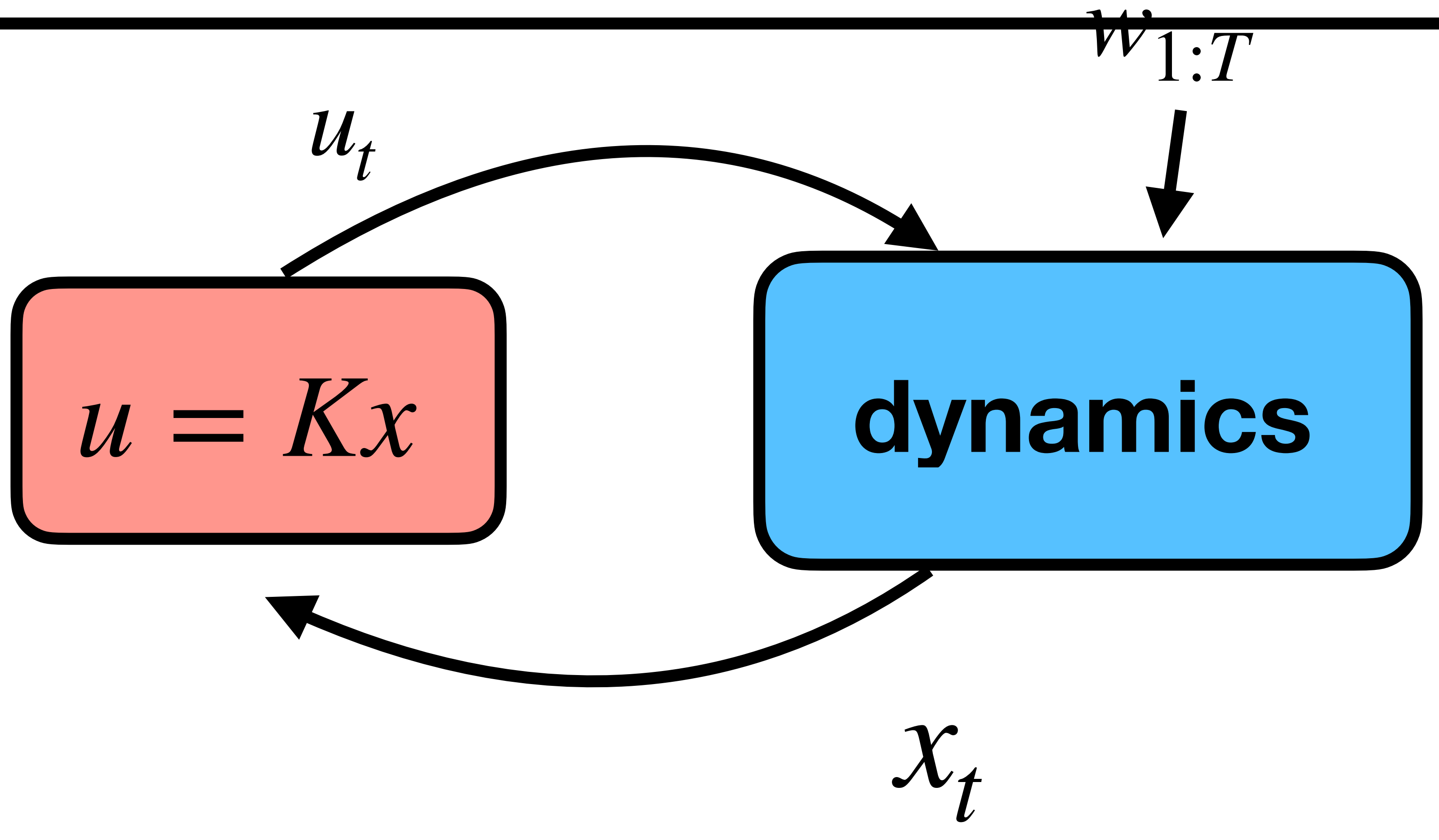

Includes optimal $\mathcal{H}_2, \mathcal{H}_\infty$ controllers

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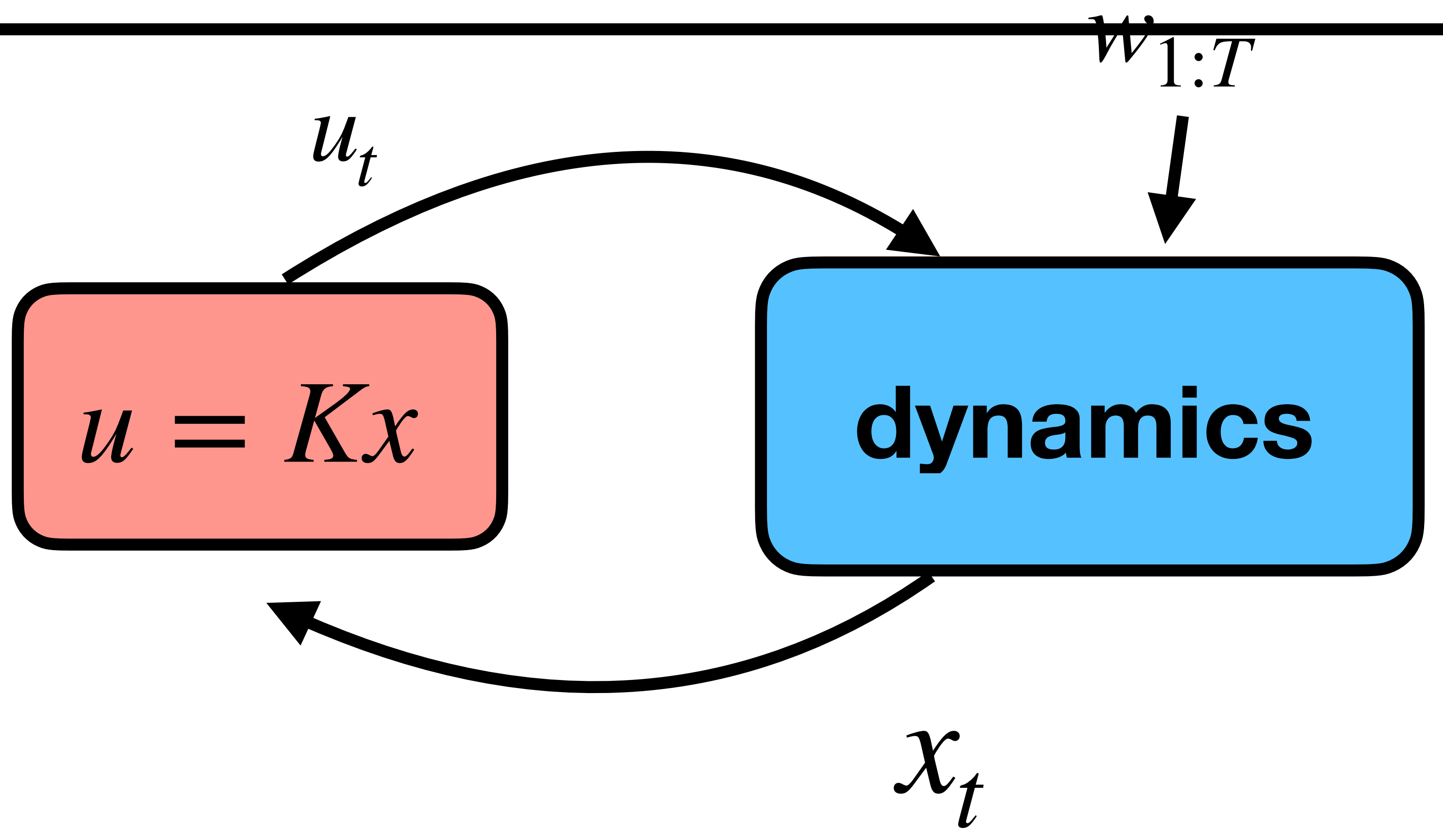

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

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Theorem: Gradient Perturbation Control (GPC) attains

$$\mathrm{Reg}_T(\mathbb{A}; \Pi_{\mathrm{feedback}}) = J_T(\mathbb{A}; W) - \inf_{\pi^K \in \Pi_{\mathrm{feedback}}} J_T(\pi^K; W) \leq \tilde{O}(\sqrt{T})$$

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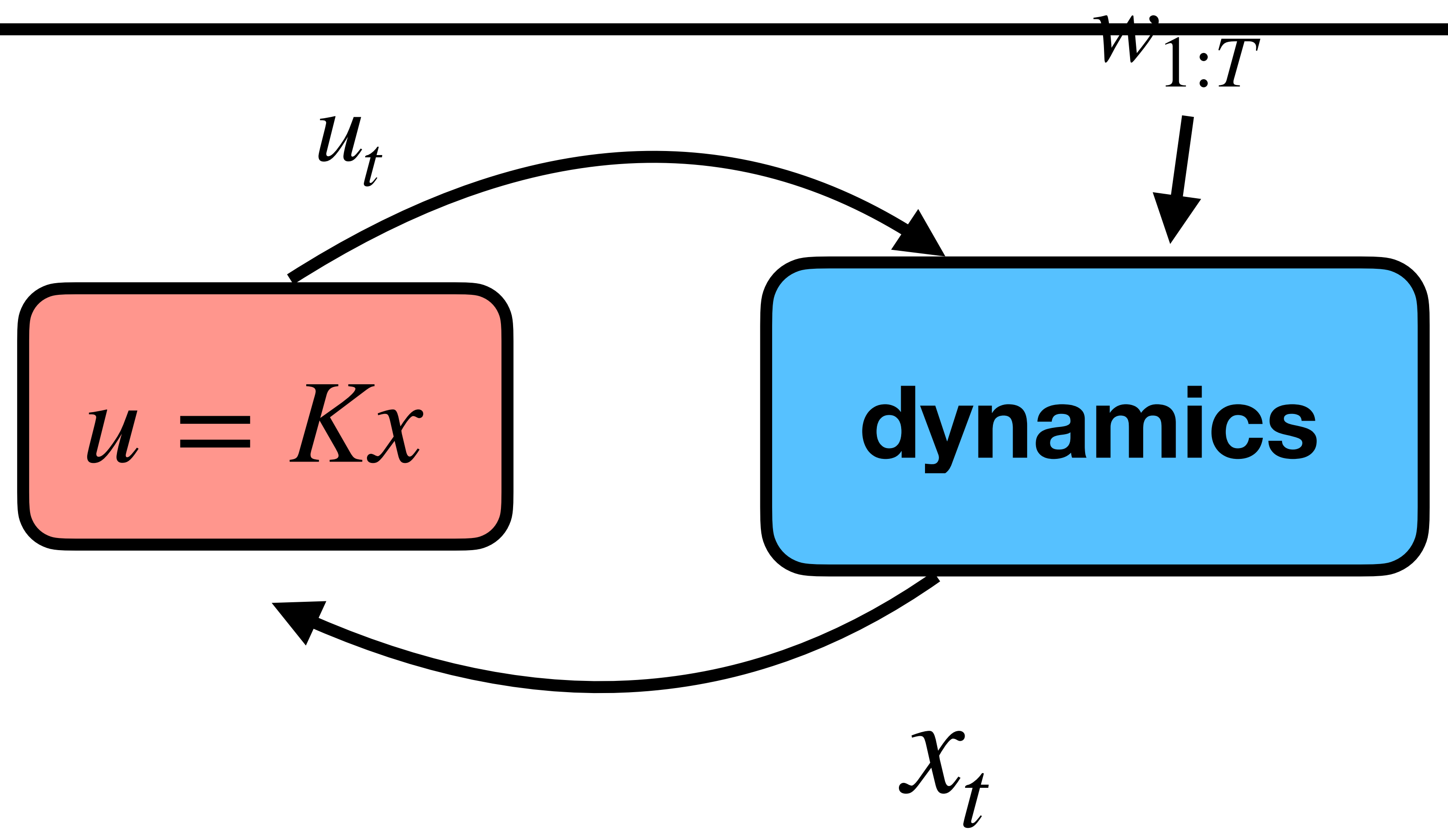

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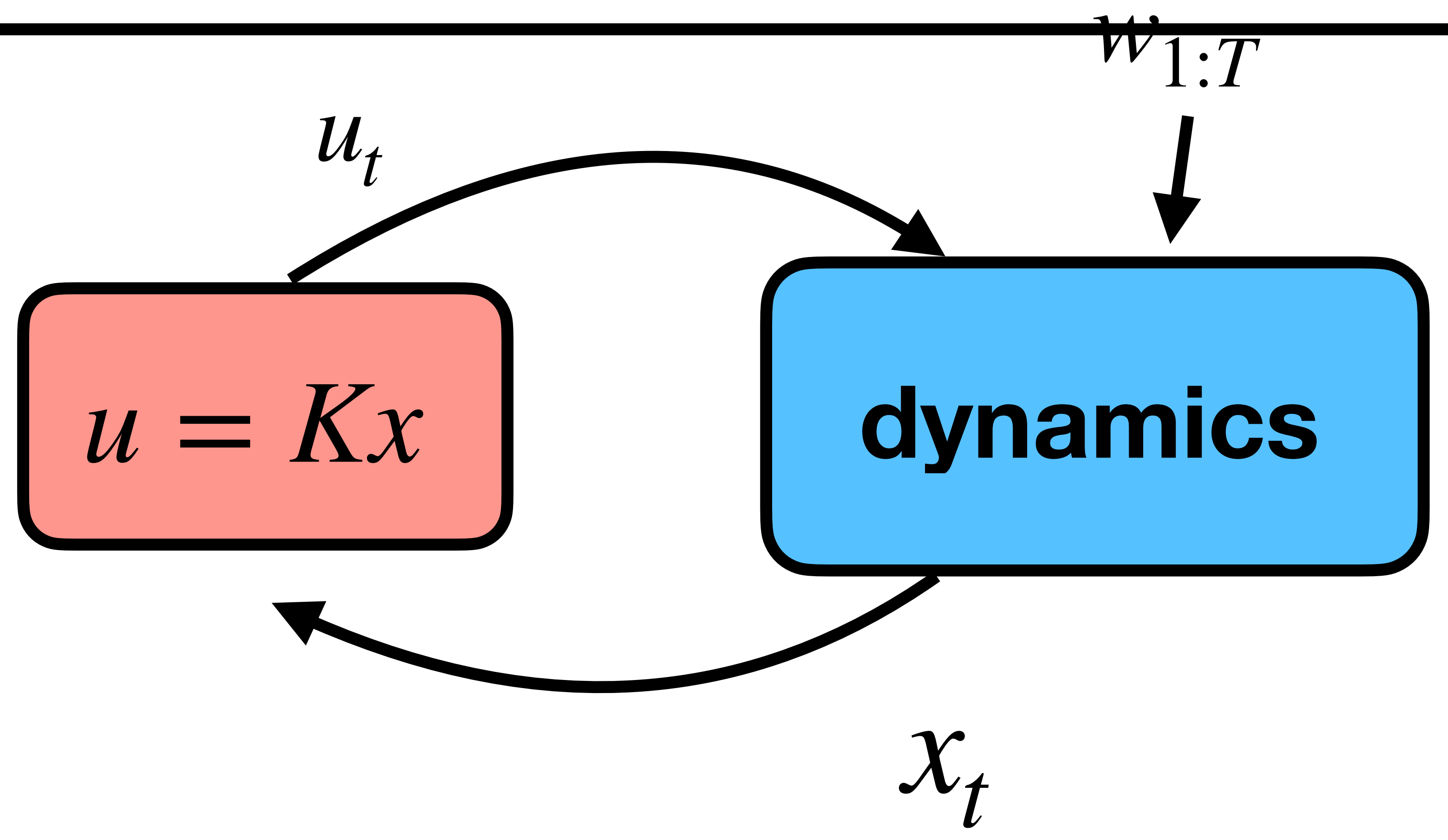

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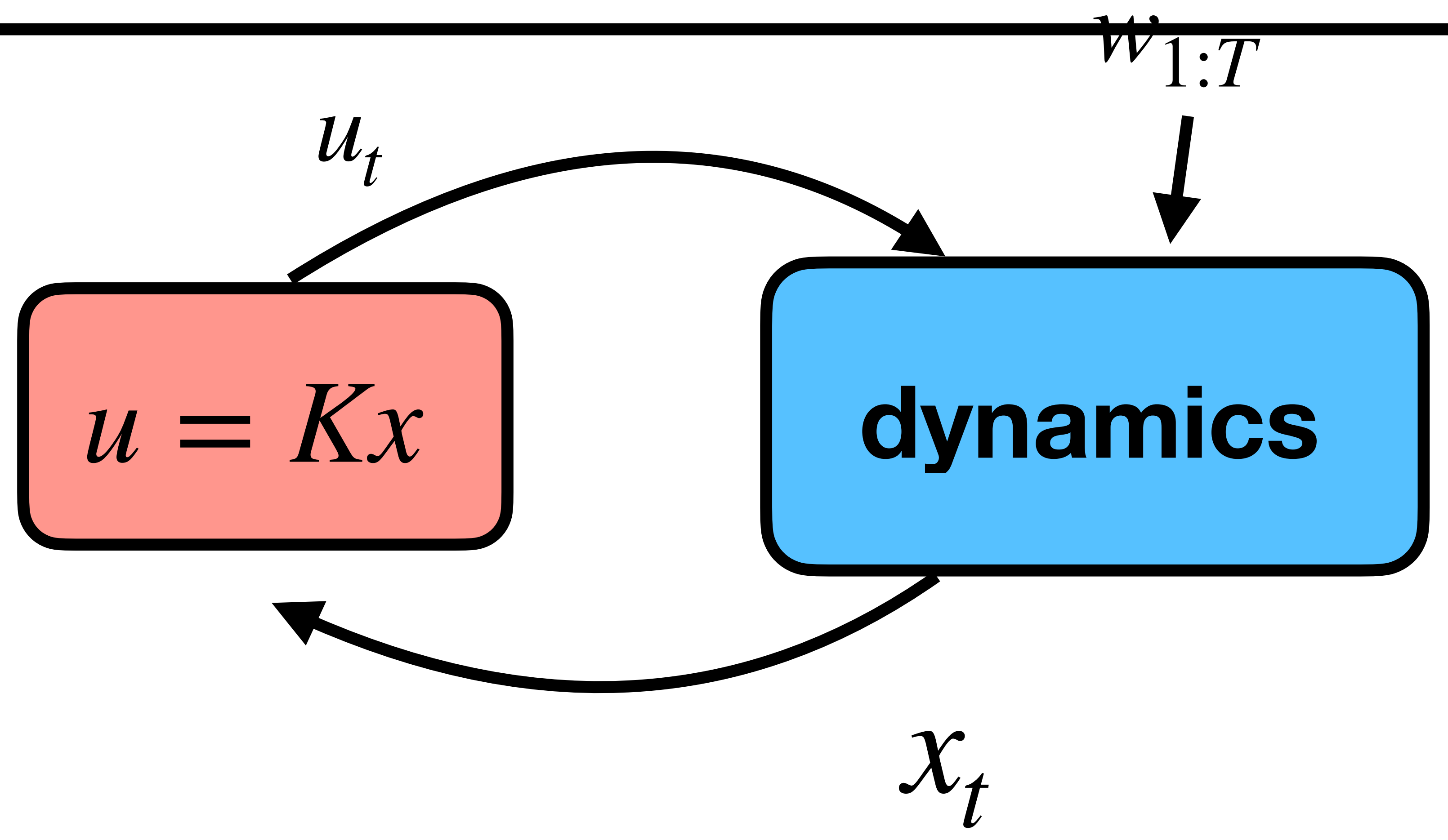

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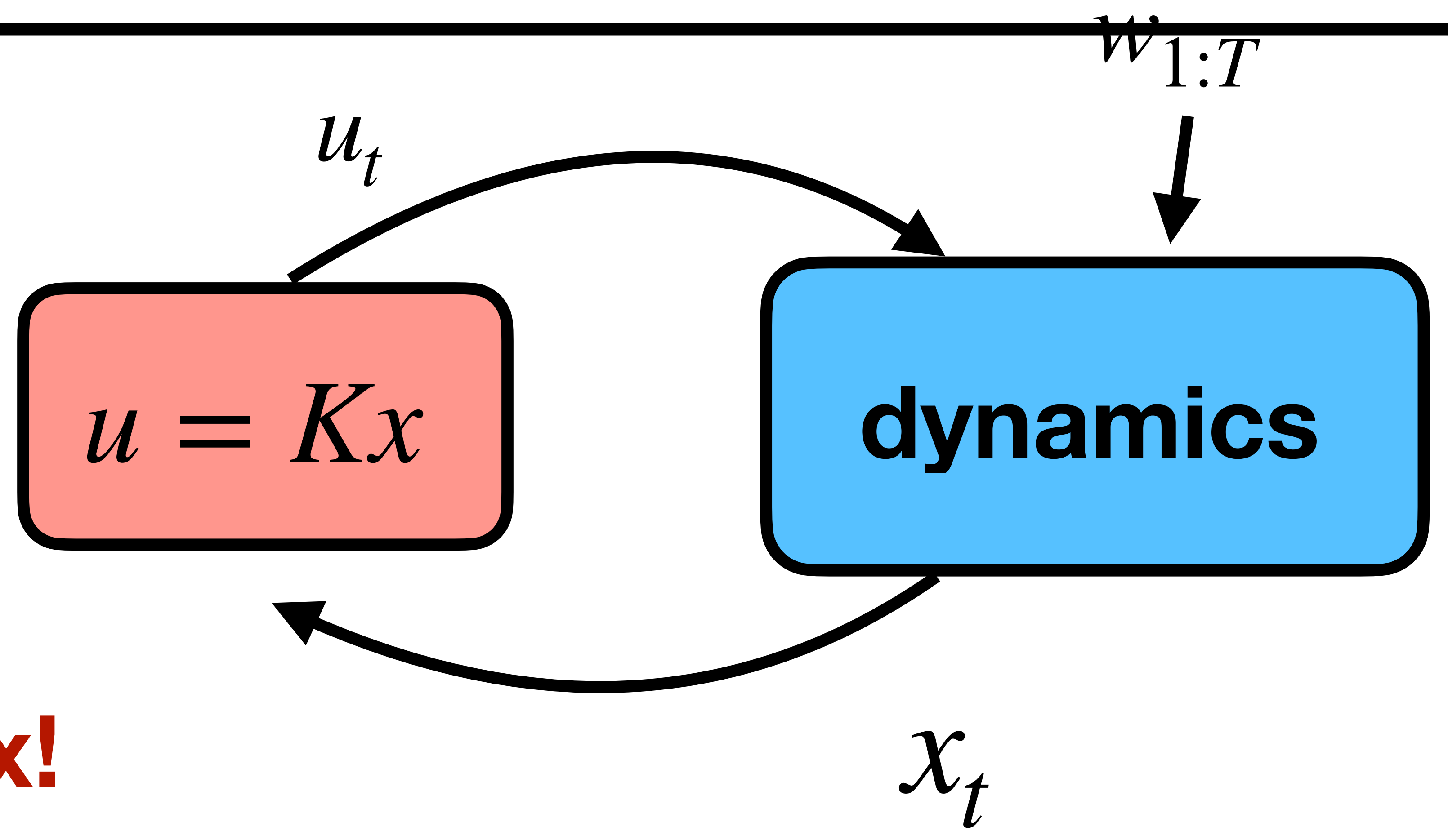

Includes optimal $\mathscr{H}_2, \mathscr{H}_\infty$ controllers **but non-convex!**

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Define: The **Disturbance Feedback Control (DFC)** parameterization:

$$\boxed{u_t^M = \sum_{i=1}^{k} M^{[i]} w_{t-i}}$$

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this is implementable **online** with known dynamics: $w_t = x_{t+1} - (Ax_t + Bu_t)$

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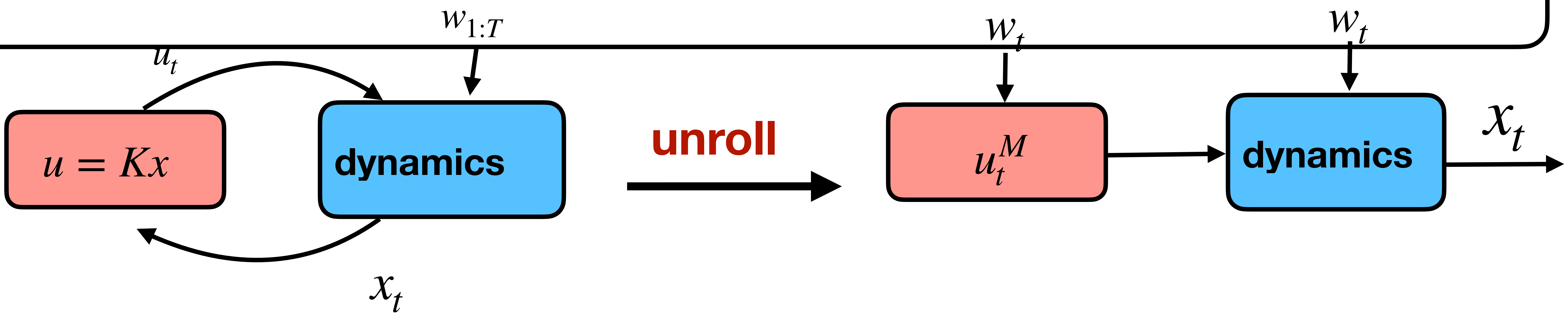

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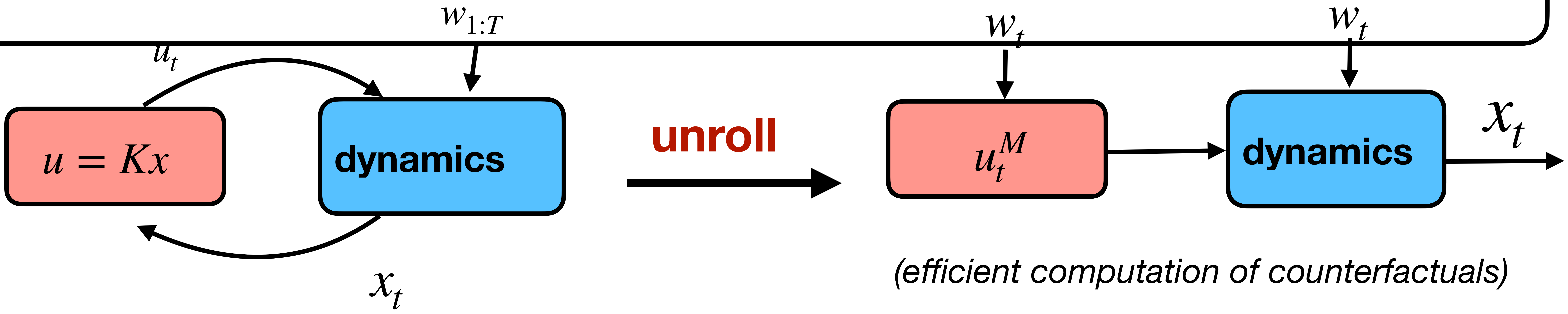

(efficient computation of counterfactuals)

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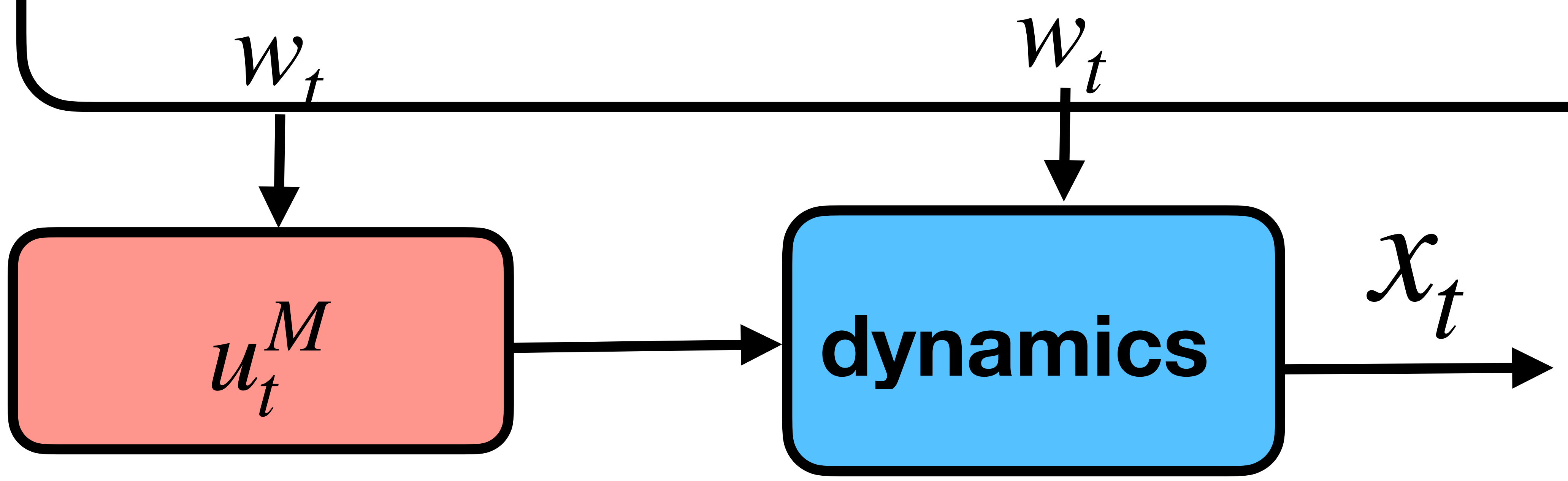
Tool 1: Convex Controller Parametrization

Observation: The mapping from $M \rightarrow (x_t^M, u_t^M)$ is **linear**

$$\boxed{u_t^M = \sum_{i=1}^{k} M^{[i]} w_{t-i}}$$

independent of past control inputs



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Tool 1: Convex Controller Parametrization

Observation: The mapping from $M \to (x_t^M, u_t^M)$ is **linear**

$$\boxed{u_t^M = \sum_{i=1}^{k} M^{[i]} w_{t-i}}$$

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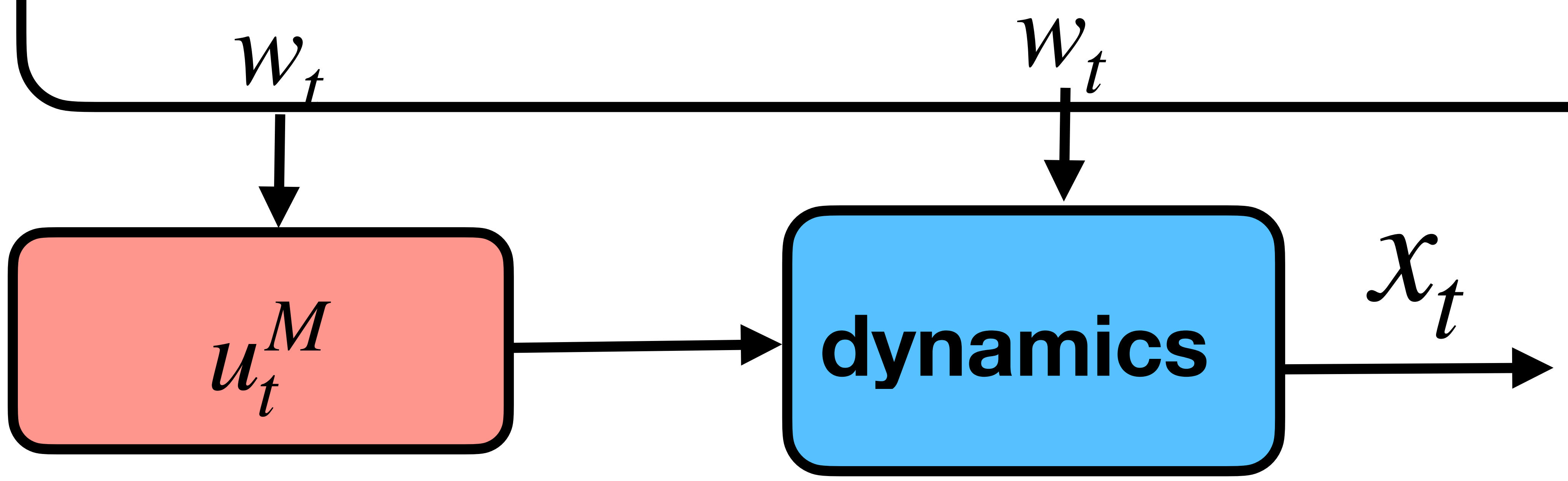

Corollary: Assuming convex costs, mapping $M \to J_T(\pi^M; W) := \sum_{t=1}^{T} c_t(x_t^M, u_t^M)$ is **convex**

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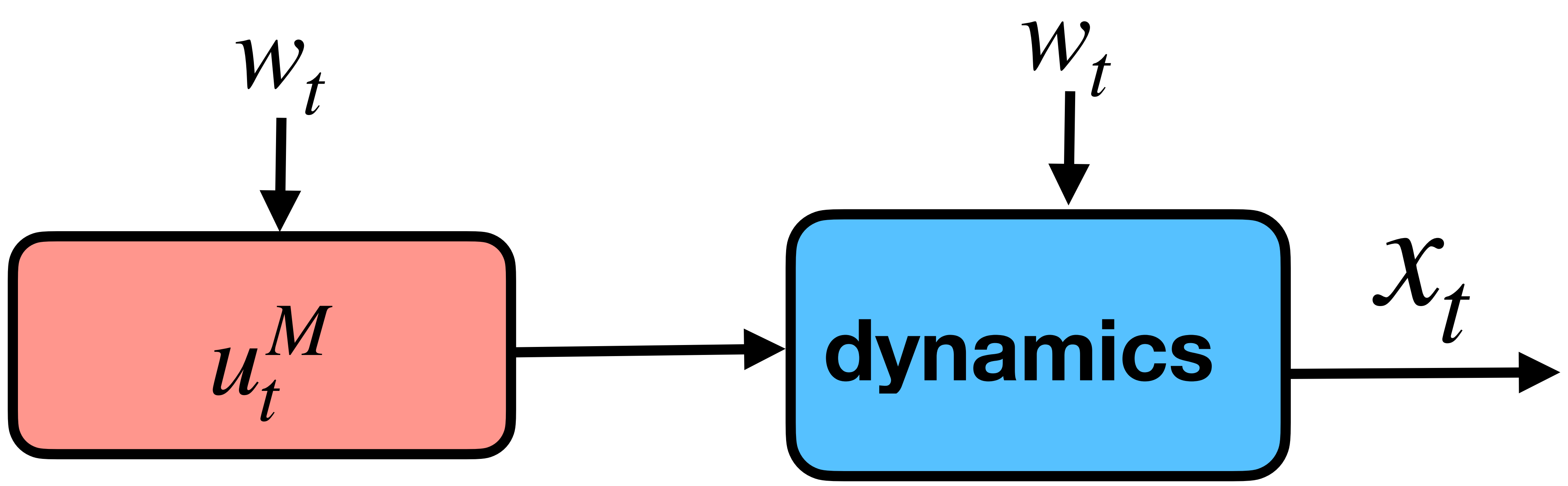
Observation: The mapping from $M \rightarrow (x_t^M, u_t^M)$ is **linear**

$$u_t^M = \sum_{i=1}^{k} M^{[i]} w_{t-i}$$

Corollary: By linearity of dynamics, mapping $M \rightarrow J_T(\pi^M; W) := \sum_{t=1}^{T} c_t(x_t^M, u_t^M)$ is **convex**



Therefore, in hindsight, we can **efficiently optimize** over controllers.

In learning theory, we call this **improper learning.**

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Tool 1: Convex Controller Parametrization

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Tool 1: Convex Controller Parametrization

Theorem: Consider any controller K such that $A + BK$ is (C, ρ) stable.

Then, $\exists$ a DFC controller $u_t^M = \sum_{i=0}^{k} M^{[i]} w_{t-i}$ with $\|M\| \le O^\star(1)$ s.t.

$\sup_t \|x_t^K - x_t^M\| \le O_\star(\rho^k)$, where $O_\star(1) = \mathrm{poly}(C, (1-\rho)^{-1})$

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states under K states under M

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Informally: DFC Controllers are an **improper relaxation** of static feedback controllers

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Tool 1: Convex Controller Parametrization

Corollary: Let Π_{feedback} denote all policies $\pi(x) = Kx$ makes s.t. $A + BK$ is (C, ρ) stable. Then, the class Π_{gpc} of all memory-k controllers with

$$\boxed{u_t^M = \sum_{i=0}^{k} M^{[i]} w_{t-i} \qquad \sum_i \|M^{[i]}\| \leq O_\star(1)}$$

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satisfies

$$\inf_M J_T(\Pi_{\text{gpc}}) - \inf_K J_T(\Pi_{\text{feedback}}) \leq O_\star(T\rho^k)$$

(assuming Lipschitz costs)

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(assuming Lipschitz costs)

suffices to optimize over Π_{gpc}

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Tool 1: Convex Controller Parametrization

Summary

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Tool 1: Convex Controller Parametrization

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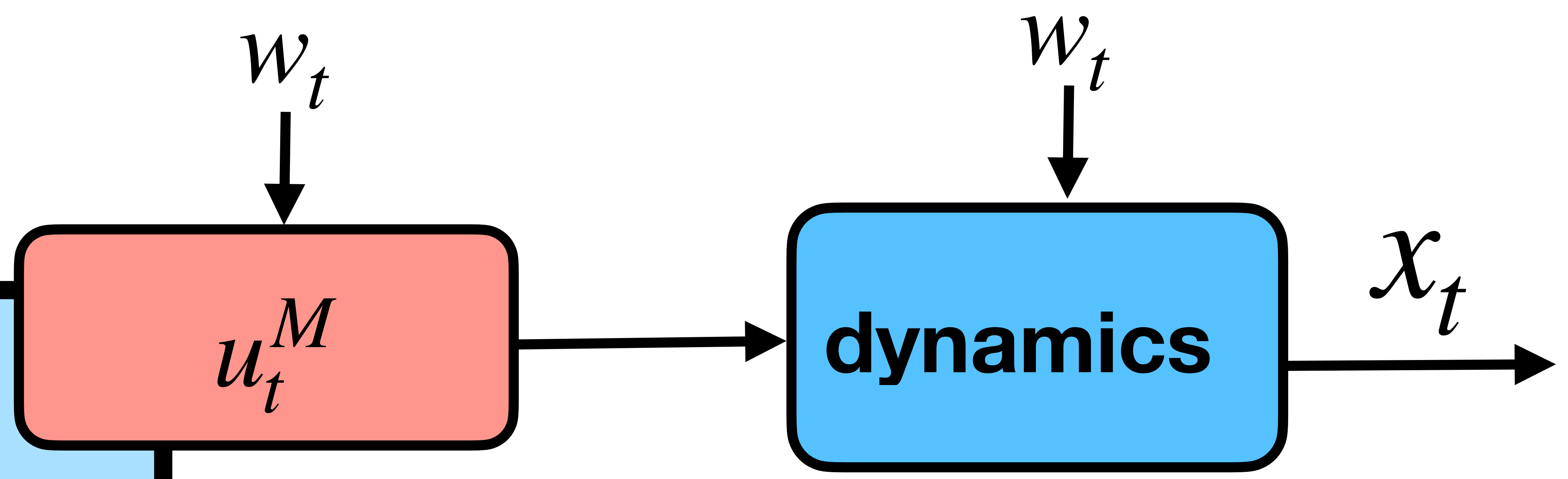

1. Efficient optimization mapping from $M \to J_T(\pi^M; W) := \sum_{t=1}^{T} c_t(x_t^M, u_t^M)$ is **convex**

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Tool 1: Convex Controller Parametrization

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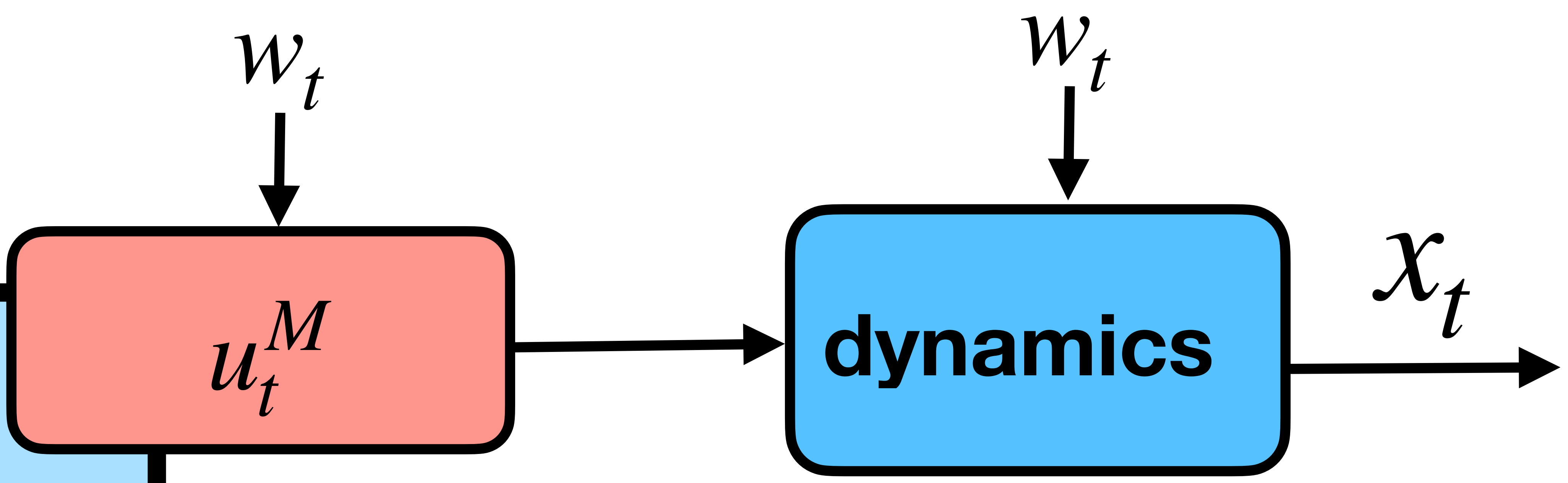

1. Efficient optimization mapping from $M \to J_T(\pi^M; W) := \sum_{t=1}^{T} c_t(x_t^M, u_t^M)$ is **convex**

2. For bounded M of memory k: $\inf_M J_T(\Pi_{\text{gpc}}) - \inf_K J_T(\Pi_{\text{feedback}}) \leq O_\star(T\rho^k)$

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The Gradient Perturbation Controller

For $t = 1,2,\ldots$

1. $\boxed{u_t \leftarrow u_t^{M_t}}$ **defined in terms of** $M = (M^{[0]}, \ldots, M^{[k]})$ ✓

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2. $\boxed{M_{t+1} \leftarrow M_t - \eta_t \nabla \tilde{F}_t(M_t)}$ **where** $\tilde{F}_t$ **is convex**

(online gradient descent)

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Tool 2: Online Convex Optimization

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Protocol: Online Convex Optimization.

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Goal: Make $\mathrm{OcoReg}_T := \sum_{t=1}^{T} f_t(\theta_t) - \inf_{\theta \in \Theta} \sum_{t=1}^{T} f_t(\theta) \ \leq o(T)$

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forces learning under adversarial uncertainty!

Tool 2: Online Convex Optimization

Algorithm: Online Gradient Optimization.

For times $t = 1, 2, \ldots,$

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Tool 2: Online Convex Optimization

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step size

Tool 2: Online Convex Optimization

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step size **gradient** (or convex subgradient)

Tool 2: Online Convex Optimization

Algorithm: Online Gradient Descent (OGD).

For times $t = 1,2,\ldots,$

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Tool 2': Reducing Online Control to OCO

Protocol: Online Control over GPC Parameterization

For times $t = 1,2,\ldots,$

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Tool 2': Reducing Online Control to OCO

Protocol: Online Control over GPC Parameterization

For times $t = 1,2,\ldots,$

Learner selects action M_t, and executes $u_t^{\mathbb{A}} = \sum_{i=0}^{k} M_t^{[i]} w_{t-i}$

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Learner suffers $c_t(x_t^{\mathbb{A}}, u_t^{\mathbb{A}})$

Dynamics evolve $x_{t+1}^{\mathbb{A}} = Ax_t^{\mathbb{A}} + Bu_t^{\mathbb{A}} + w_t$

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Tool 2’: Reducing Online Control to OCO

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Tool 2': Reducing Online Control to OCO

1. $c_t(x_t^{\mathbb{A}}, u_t^{\mathbb{A}})$ = **true algorithm cost**

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counterfactual cost with memory k

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counterfactual cost with memory k

Specifically $\tilde{F}_t(M) = F_t(M, \ldots, M) = c_t\left(\sum_{i=1}^{k} A^{i-1} B u_{t-i}^{M}, u_t^{M}\right)$

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This is convex in M!

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counterfactual cost with memory k

Update $M_{t+1} \leftarrow M_t - \eta \nabla \tilde{F}_t(M_t)$

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Update $M_{t+1} \leftarrow M_t - \eta \nabla \tilde{F}_t(M_t)$ **Online Gradient Descent**

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⚠️⚠️**Warning: Technical Part**⚠️⚠️

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$$\mathrm{Reg}_T(\mathbb{A}; \Pi_{\text{feedback}}) = J_T(\mathbb{A}; W) - \inf_{\pi^K \in \Pi_{\text{feedback}}} J_T(\pi^M; W)$$

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Online Convex Optimization with Memory

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$$\boxed{= \sum_{t=1}^T F_t(M_t, \ldots, M_{t-k}) - \inf_M \sum_{t=1}^T F_t(M, \ldots, M) + O_\star(T\rho^k)}$$

Online Convex Optimization with Memory **stability**

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Tool 2': Reducing Online Control to OCO

$$\mathrm{Reg}_T(\mathbb{A};\Pi_{\text{feedback}}) \le \sum_{t=1}^{T} F_t(M_t,\ldots,M_{t-k}) - \inf_M \sum_{t=1}^{T} F_t(M,\ldots,M) + O_\star(T\rho^k)$$

Algorithm: Gradient-Perturbation Controller (GPC)

$$M_{t+1} = M_t - \eta_t \nabla \tilde{F}_t(M_t) \qquad \tilde{F}_t(M) = F_t(M,\ldots,M) \qquad u_t \leftarrow u_t^M$$

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2. **Take gradient updates as if you M_t was not changing.**

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Tool 2': Reducing Online Control to OCO

$$\mathrm{Reg}_T(\mathbb{A}; \Pi_{\mathrm{feedback}}) \le \sum_{t=1}^T F_t(M_t, \ldots, M_{t-k}) - \inf_M \sum_{t=1}^T F_t(M, \ldots, M) + O_\star(T\rho^k)$$

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Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

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Intuition: Combine the standard regret for OCO with bound that

$$|F_t(M_t, \ldots, M_{t-k}) - \tilde{F}_t(M)| \le O_\star(1) \cdot \sum_{1 \le \ell, j, \le k} \eta_{t-i} \le k^2 \eta_{t-k} \lesssim O_\star(k^2\sqrt{T})$$

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Corollary: If $\eta_t = O(1/\sqrt{t}), k \gg \log T$

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$$\mathrm{Reg}_T(\mathbb{A};\Pi_{\mathrm{feedback}}) \leq O_\star(k^2\sqrt{T} + T\rho^k) = \tilde{O}(\sqrt{T})$$

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finally! we are done :)

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Summary: Gradient Perturbation Controller

For $t = 1,2,\ldots$

1. $\boxed{u_t \leftarrow u_t^{M_t}}$ **defined in terms of** $M = (M^{[0]}, \ldots, M^{[k]})$ ✓

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

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Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

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Theorem: **Gradient Perturbation Control** (GPC) attains

$$\boxed{\mathrm{Reg}_T(\mathbb{A}; \Pi_{\text{feedback}}) = J_T(\mathbb{A}; W) - \inf_{\pi^K \in \Pi_{\text{feedback}}} J_T(\pi^M; W) \leq \tilde{O}(\sqrt{T})}$$ ✓

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From Stable to Stabilized

Previously, we assumed **stable dynamics:** $\|A^s\| \leq C\rho^s$

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

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Previously, we assumed **stable dynamics:** $\|A^s\| \leq C\rho^s$

Here, we assume we know **any** K_0 such that: $\|(A + BK_0)^s\| \leq C\rho^s$ **is closed-loop stable**

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**stay tuned for if you don't know* K_0

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Theorem: GPC with $u_t \leftarrow K_0 x_t + \sum_{i=0}^{k} M^{[i]} w_{t-i}$ attains

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$\mathrm{Reg}_T(\mathbb{A}; \Pi_{\mathrm{feedback}}) \leq \tilde{O}(\sqrt{T})$ **Proof:** Same, but fold K_0 into dynamics

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Summary

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Summary

1. We introduce and analyze the **Gradient Perturbation Controller** (GPC)

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1. We introduce and analyze the **Gradient Perturbation Controller** (GPC)

2. It is built on **Disturbance Feedback Control** (DFC) as convex, “improper” representation of linear controllers (equivalent to SLS, *Anderson et al. ’19*)

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Summary

1. We introduce and analyze the **Gradient Perturbation Controller** (GPC)

2. It is built on **Disturbance Feedback Control** (DFC) as convex, "improper" representation of linear controllers (equivalent to SLS, *Anderson et al. '19*)

3. We build on the **Online Convex Optimization** (OCO) framework to develop a gradient-based controller

Agrawal, Bullins, Hazan, Kakade, Singh "Online Control with Adversarial Disturbances", 2019

Generalizations

Roadmap

2. Nature's Y's: Partially Observed, Known-Dynamics
3. Unknown Dynamics: System Identification
4. Optimal Regret: Leveraging Curvature

Roadmap

2. Nature’s Y’s: Partially Observed, Known-Dynamics

From Full Observation to Nature's Y's

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Goal: Compete the linear controllers for **partially observed** $y_t = Cx_t + e_t$

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From Full Observation to Nature's Y's

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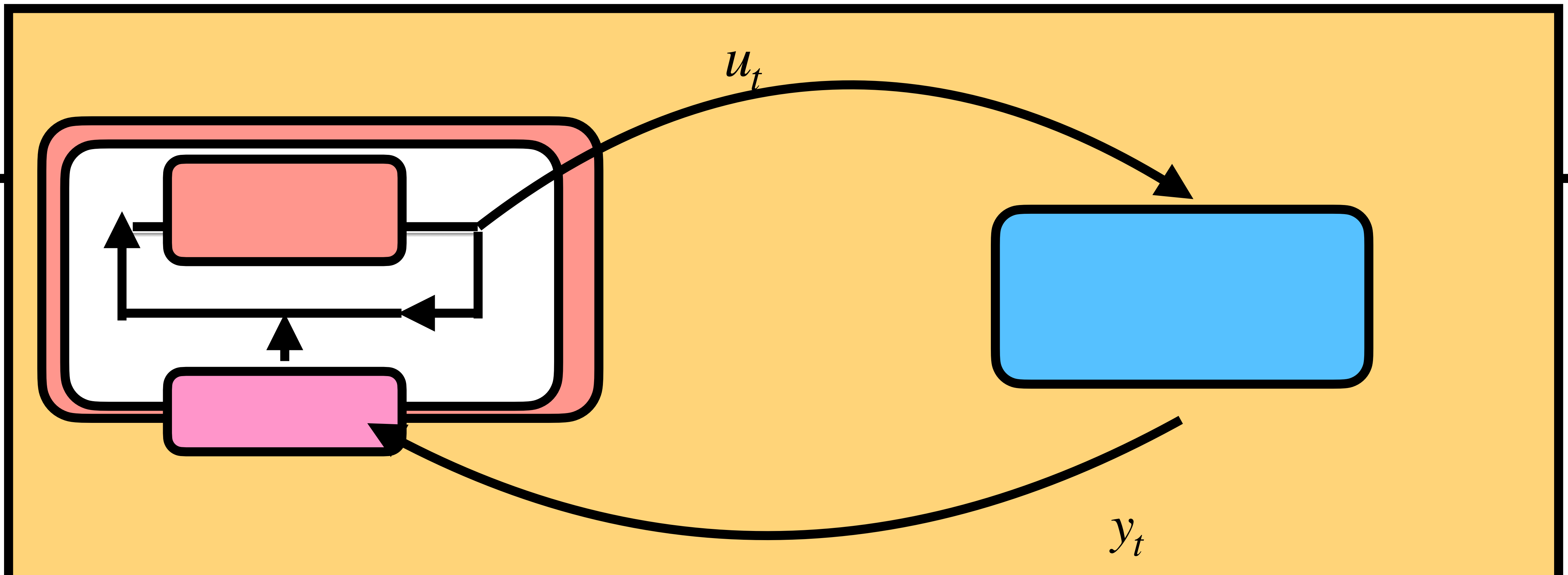
Challenge 1: GPC controller needed to "see" w_t, which are now **hidden**

Challenge 2: Static feedback on y_t, $u_t = Ky_t$, is **suboptimal** for partial observation.



$$z_{t+1} = A_\pi z_t + B_\pi y_t$$

$$u_t = C_\pi z_t + D_\pi y_t$$

From Full Observation to Nature's Y's

Idea: Convex parametrization (control lang.) or improperness (learning lang.)

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

From Full Observation to Nature's Y's

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Define the **Markov Operators** $y_t = e_t + \sum_{i=0}^{t} G_{w \to y}^{[i]} w_{t-i} + G_{u \to y}^{[i]} u_{t-i}$

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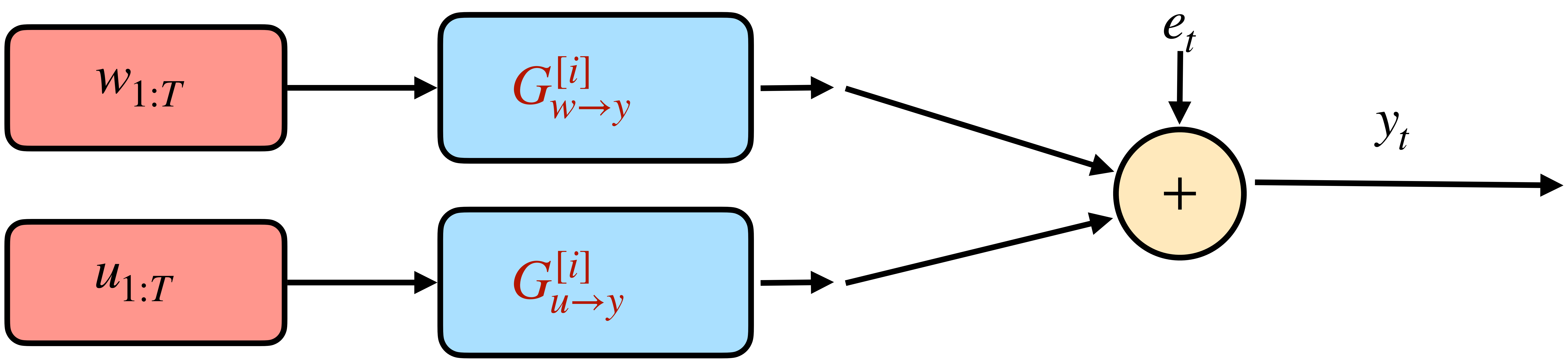

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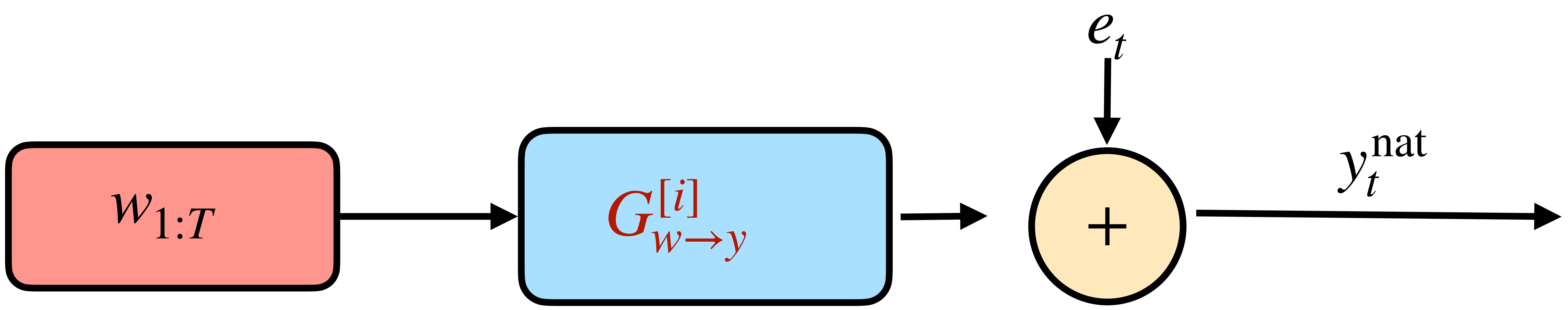

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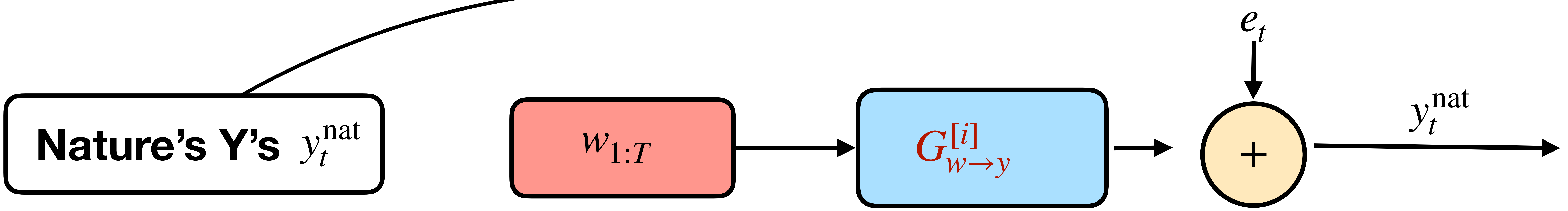

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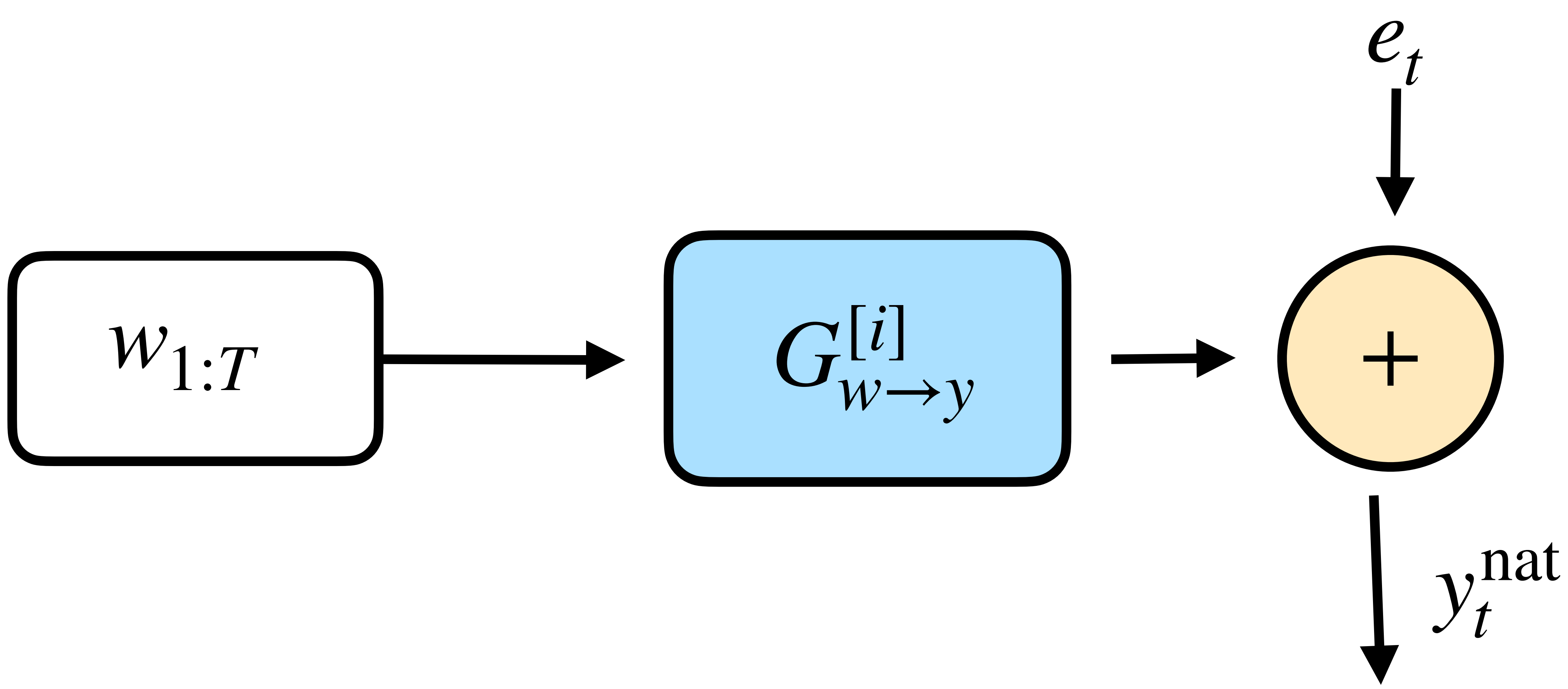
From Full Observation to Nature's Y's



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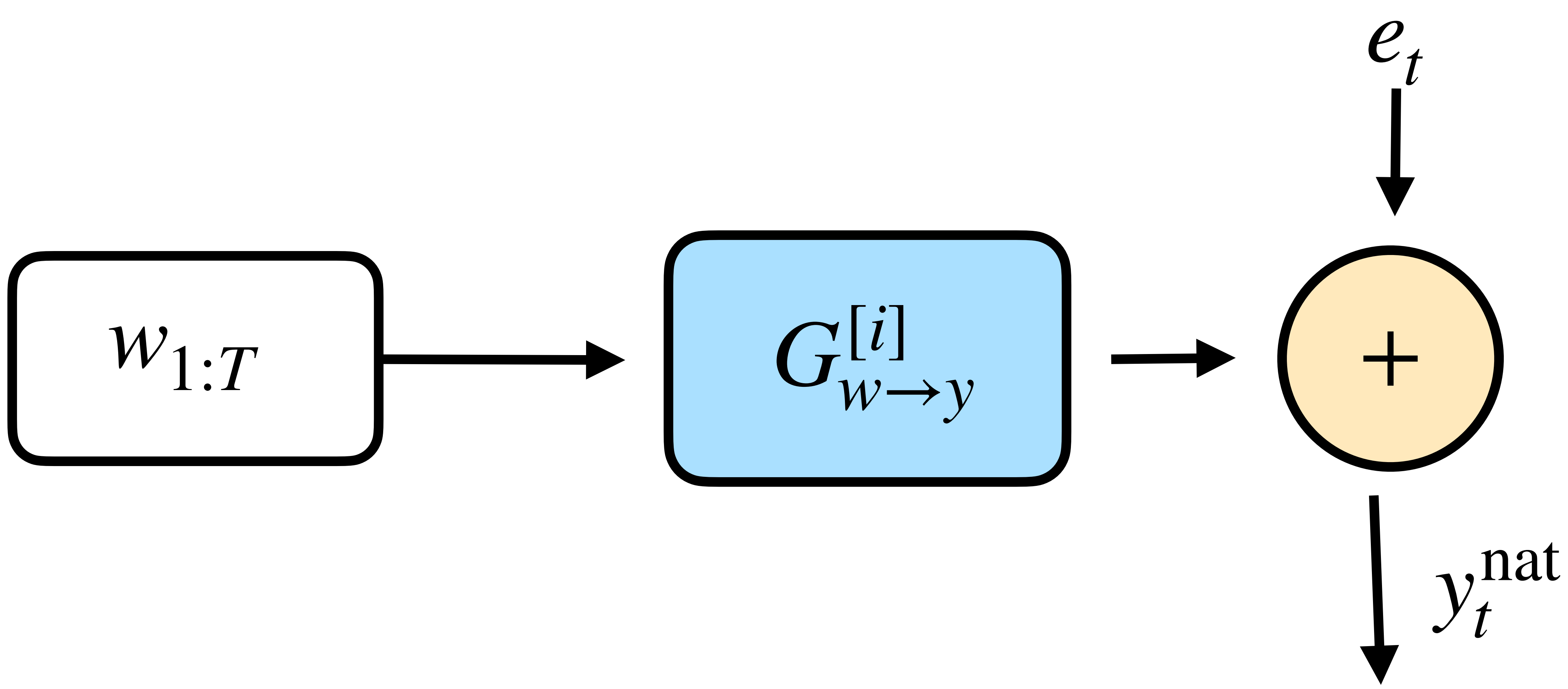
From Full Observation to Nature's Y's

Definition: Disturbance Feedback Control (DFC) $u_t^M = \sum_{i=0}^{t} M^{[i]} y_{t-i}^{\text{nat}}$



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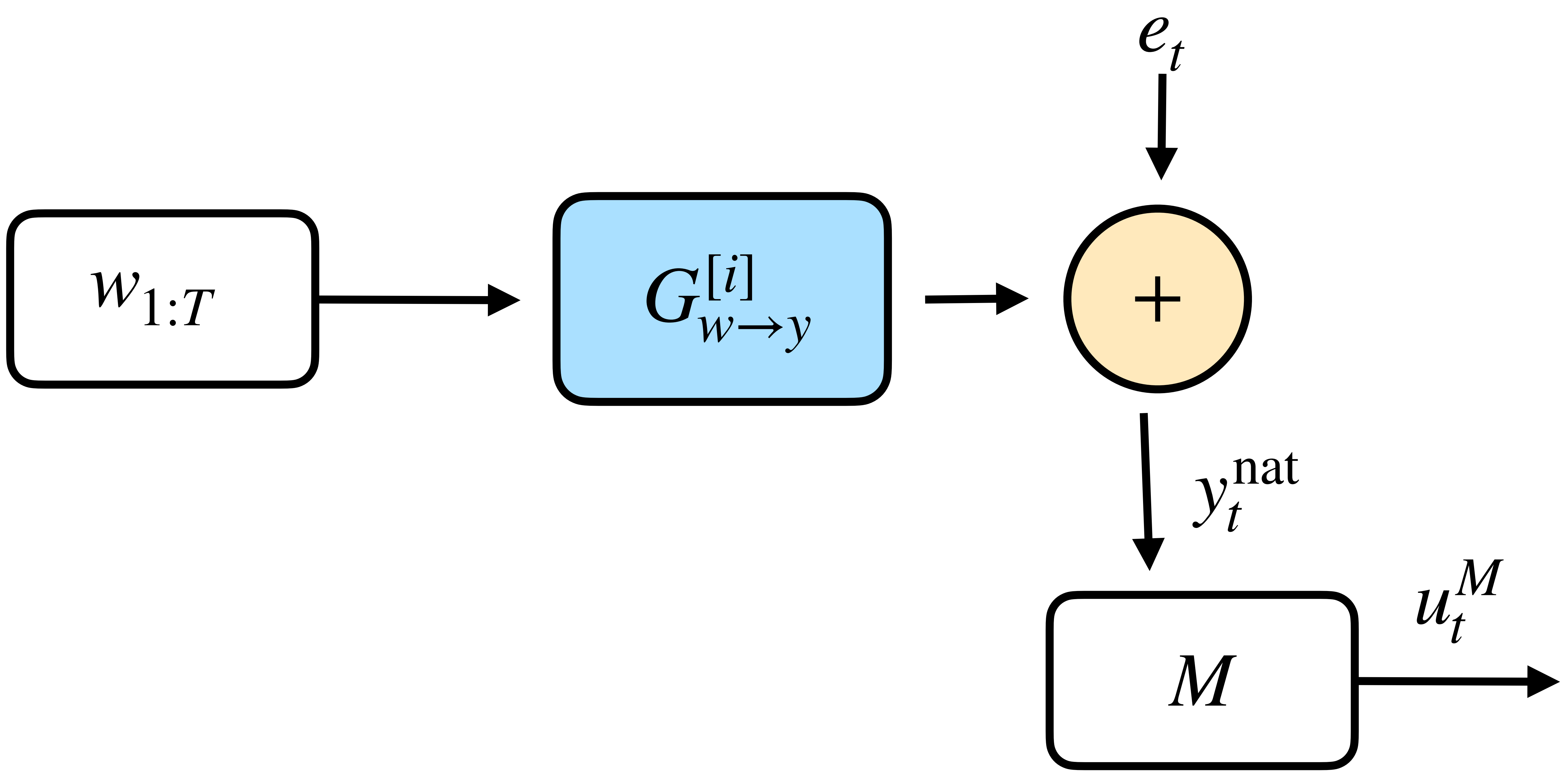

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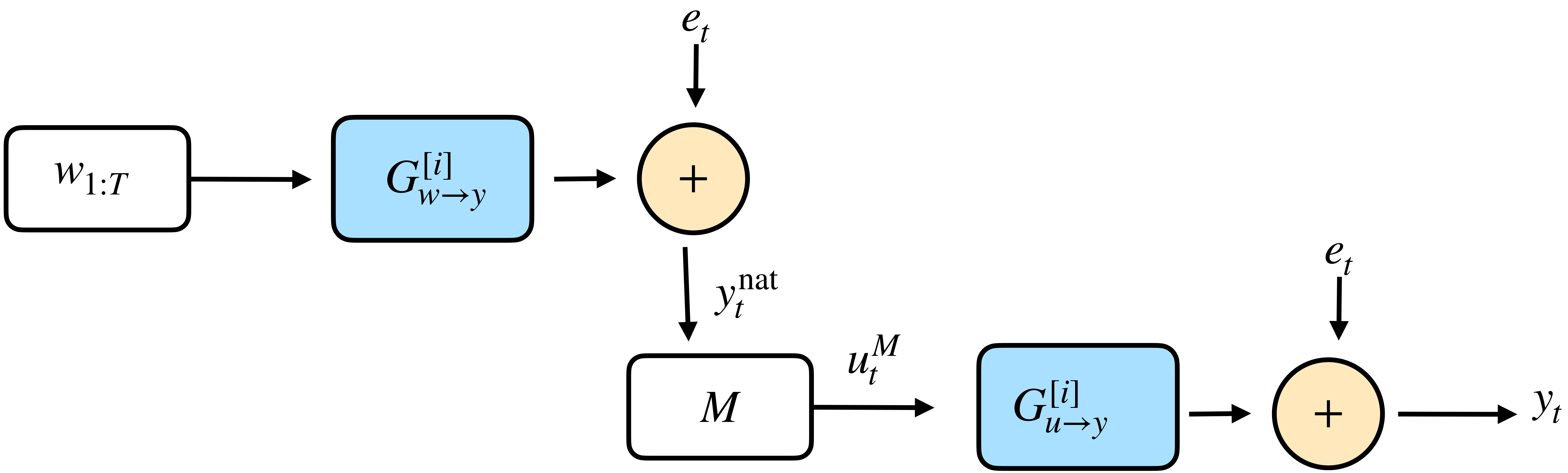

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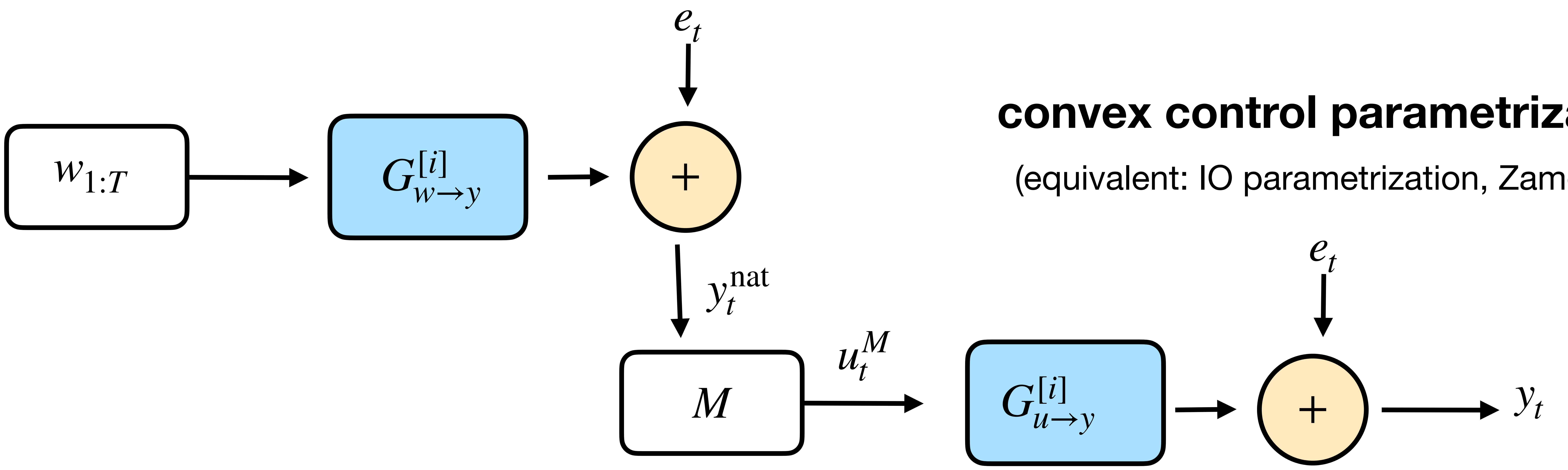

convex control parametrization!

(equivalent: IO parametrization, Zames '81)

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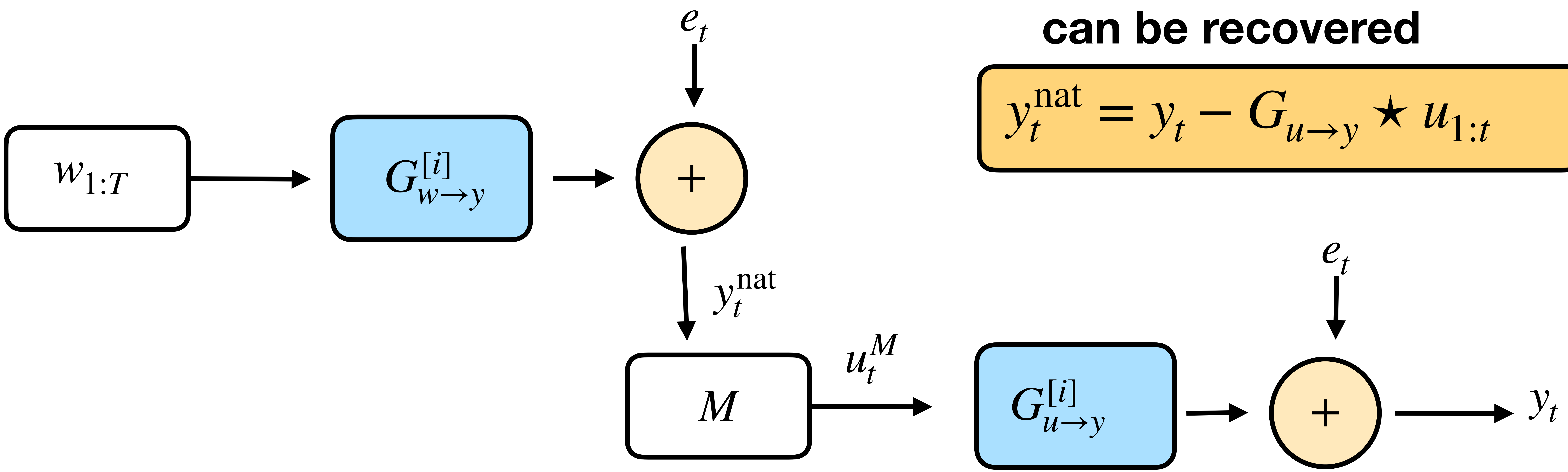
From Full Observation to Nature's Y's

Definition: Disturbance Feedback Control (DFC) $u_t^M = \sum_{i=0}^{t} M^{[i]} y_{t-i}^{\text{nat}}$

can be recovered

$$y_t^{\text{nat}} = y_t - G_{u\to y} \star u_{1:t}$$



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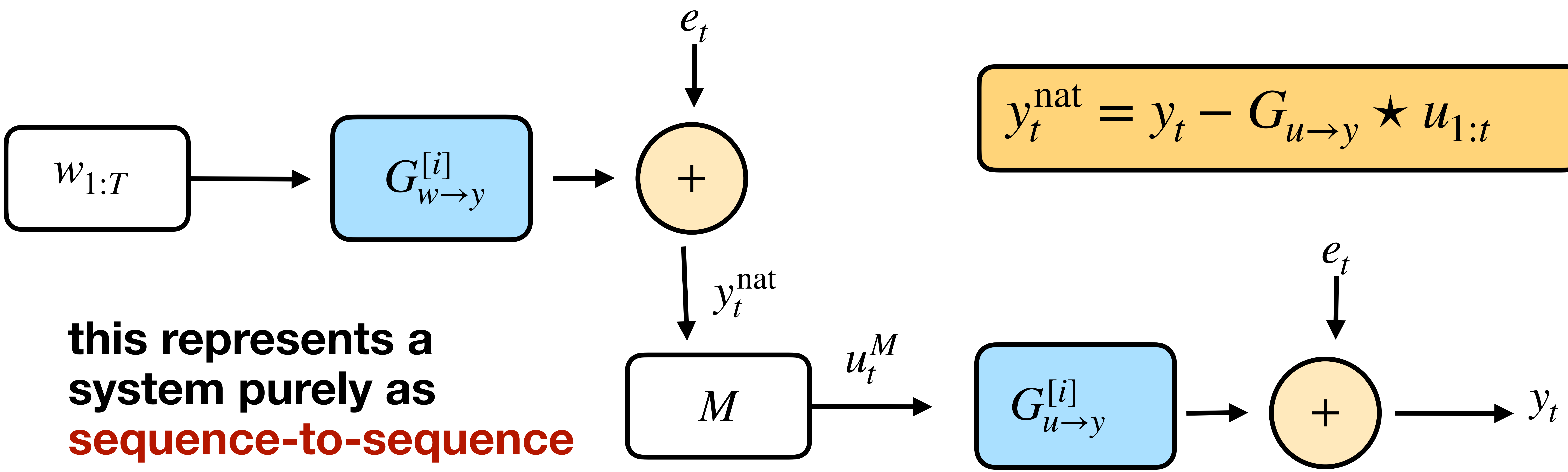

this represents a system purely as sequence-to-sequence

(e.g. Sutskever, Vinyals, Le)

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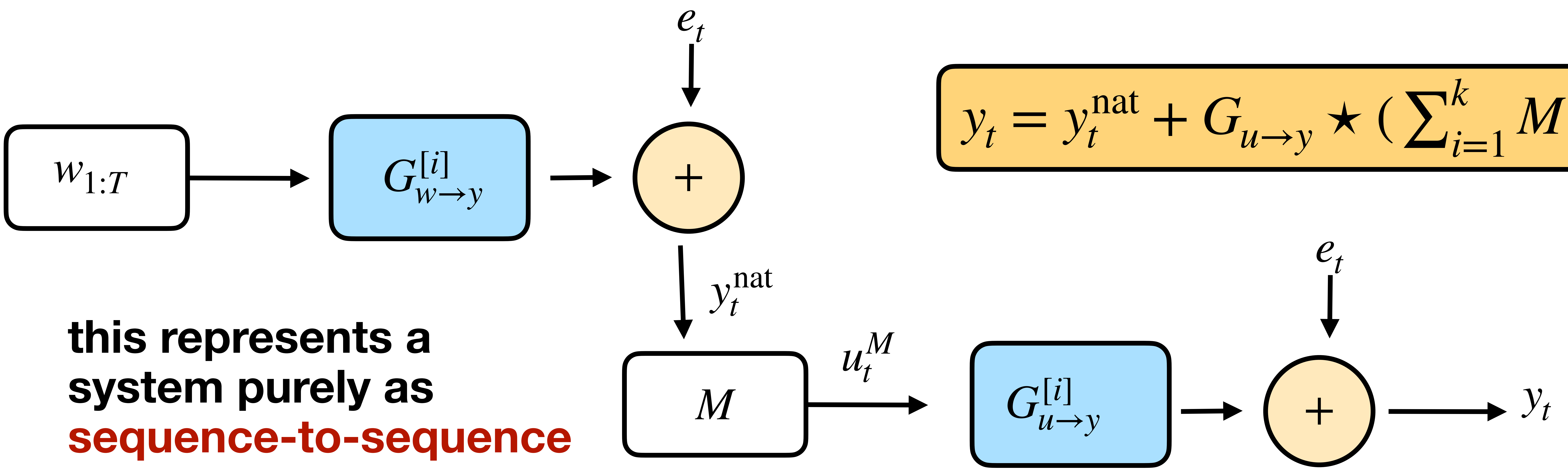

$$y_t = y_t^{\text{nat}} + G_{u\to y} \star \left(\sum_{i=1}^{k} M^{[i]} y_i^{\text{nat}}\right)$$

this represents a system purely as sequence-to-sequence

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From Full Observation to Nature's Y's

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From Full Observation to Nature's Y's

Assume **Markov Operators*** are (C, ρ)**-stable:** $\max\{\|G^{[i]}_{w\to y}\|, \|G^{[i]}_{u\to y}\|\} \le C\rho^i$

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

From Full Observation to Nature's Y's

Assume **Markov Operators*** are (C,ρ)-**stable:** $\max\{\|G^{[i]}_{w\to y}\|, \|G^{[i]}_{u\to y}\|\} \le C\rho^i$

Theorem (Nature's Y's)**:** Any stabilizing, **dynamic** linear controller can be approximated by the **Disturbance Response Control** (DRC)

$$u^M_t = \sum_{i=0}^{t} M^{[i]} y^{\mathrm{nat}}_{t-i} \qquad \sum_i \|M^{[i]}\| \le O_\star(1)$$

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

From Full Observation to Nature's Y's

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Theorem (Nature's Y's Regret)**:** Online gradient descent with the **Disturbance Response Control** (DRC)

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Theorem (Nature's Y's Regret)**:** Online gradient descent with the **Disturbance Response Control** (DRC)

$$u_t = \sum_{i=0}^{t} M_t^{[i]} y_{t-i}^{\text{nat}} \qquad M_{t+1} = M_t - \eta_t \nabla \tilde{F}_t(M_t)$$

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

From Full Observation to Nature's Y's

Assume **Markov Operators*** are (C, ρ)**-stable:** $\max\{\|G^{[i]}_{w\to y}\|, \|G^{[i]}_{u\to y}\|\} \leq C\rho^i$

Theorem (Nature's Y's Regret)**:** Online gradient descent with the **Disturbance Response Control** (DRC)

$$u_t = \sum_{i=0}^{t} M_t^{[i]} y_{t-i}^{\text{nat}} \qquad M_{t+1} = M_t - \eta_t \nabla \tilde{F}_t(M_t)$$

obtains $$\inf_M \mathrm{Reg}_T(\mathbb{A}; \Pi_{\text{drc}}) \leq \tilde{O}(\sqrt{T})$$

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Generalizes to known stabilizing controller (eg. LQG) via Youla-Kućera Par.

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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The entire algorithm can be defined using Markov operators (Improper)

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

Summary

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1. We study **partial observability** ($y_t = Cx_t + e_t$)

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2. We introduce and analyze the **Nature's Y's parameterization** (DFC)

3. We show that the same rate of regret is achievable with essentially **the same principles.**

Roadmap

3. **Unknown Dynamics**: System Identification

From Known to Unknown Dynamics

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

From Known to Unknown Dynamics

Goal: Compete the linear controllers even if (A, B, C) **unknown**

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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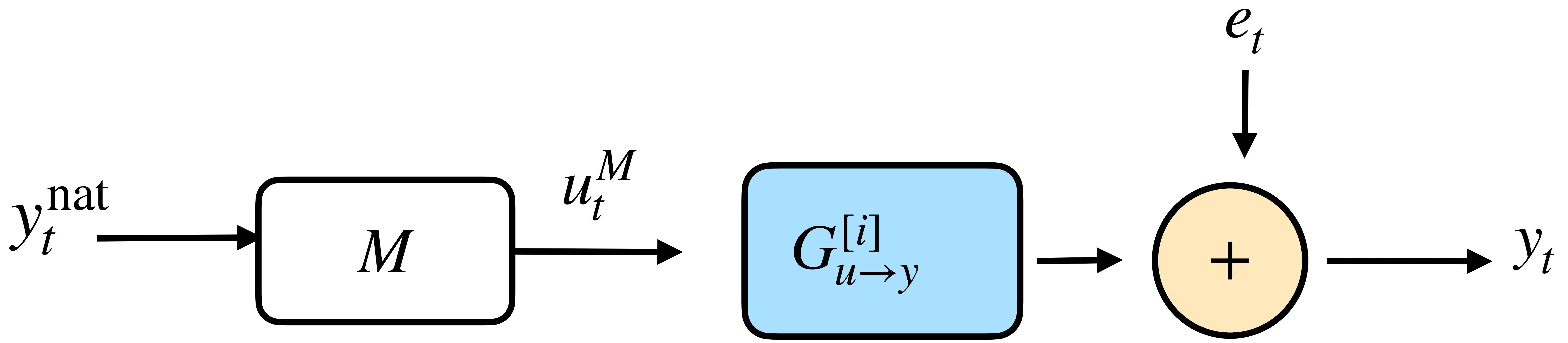

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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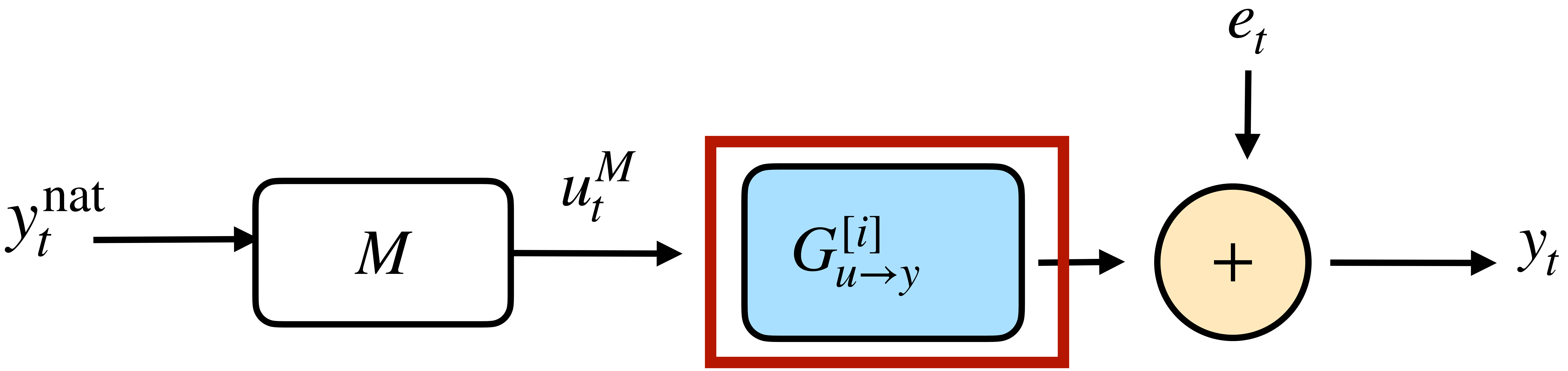

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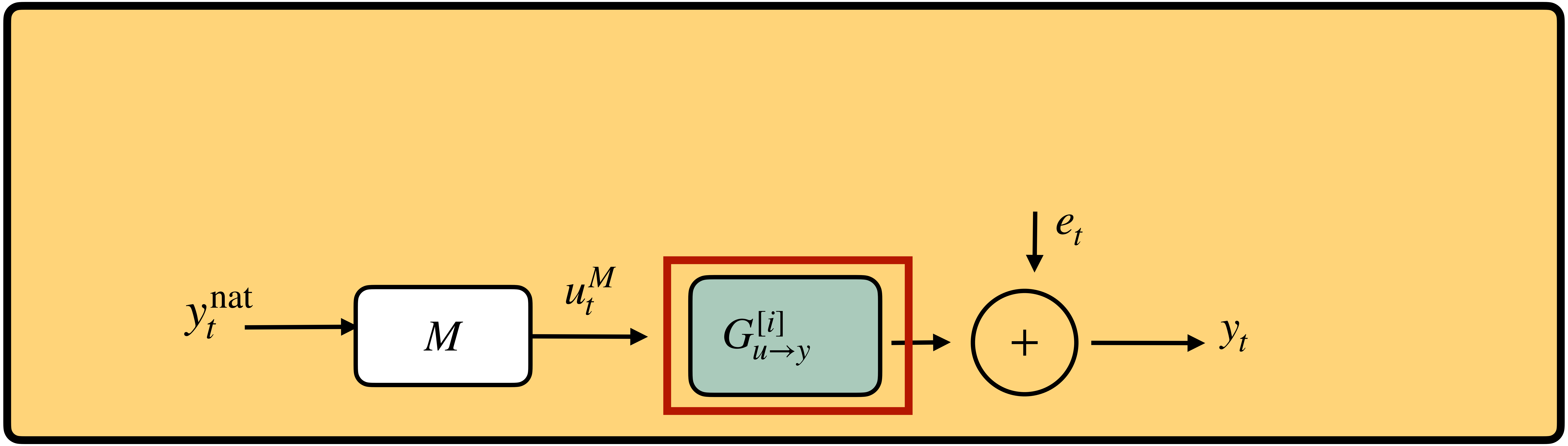

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

From Known to Unknown Dynamics

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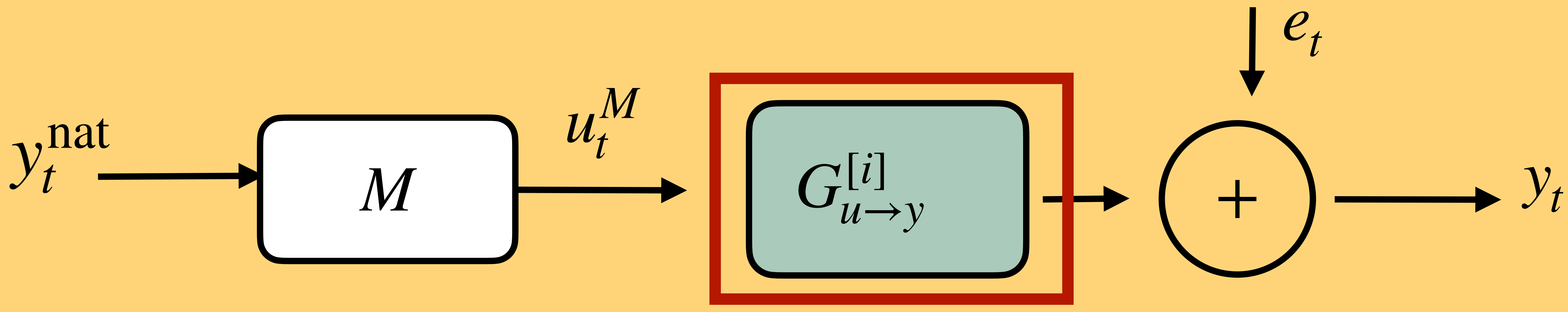

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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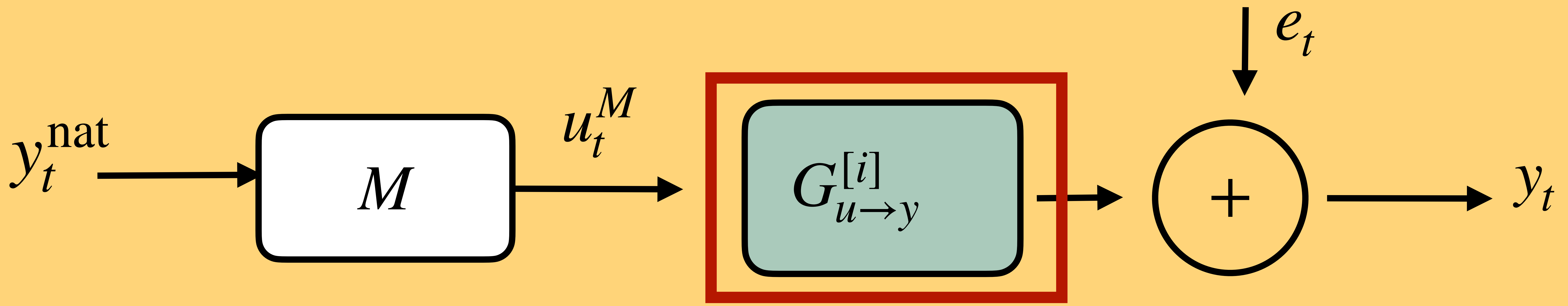

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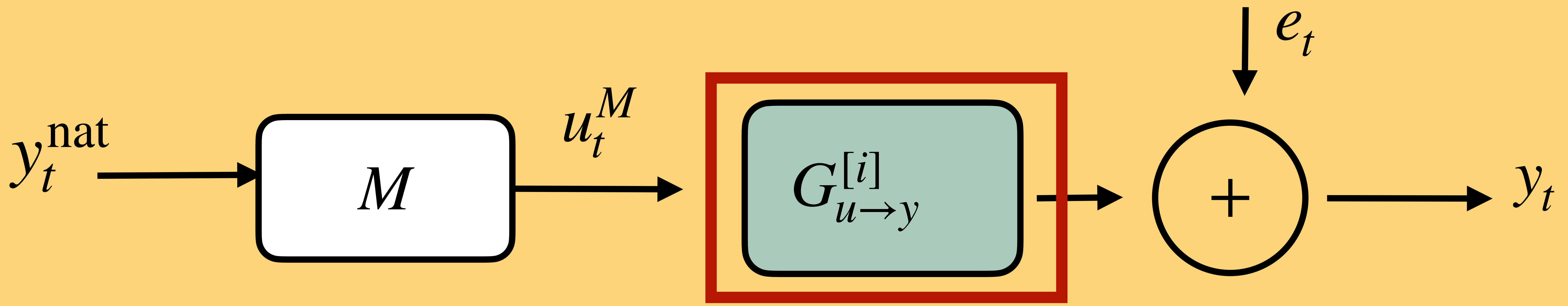

Proposition: $\mathrm{Reg}_T \leq \tilde{O}(1)\left(\sqrt{T} + T\|\hat{G}_{\mathrm{ls}} - G\| + T_0\right)$

known regret **cost for error** **cost for estimation**

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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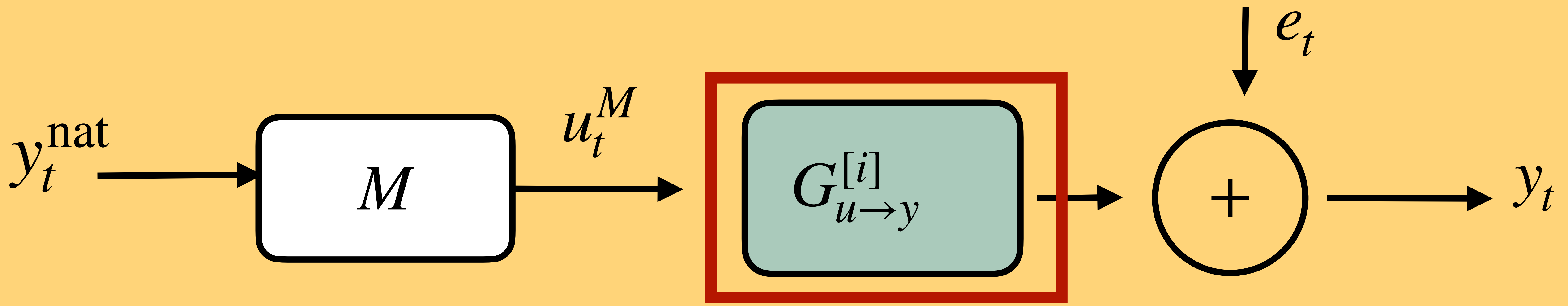

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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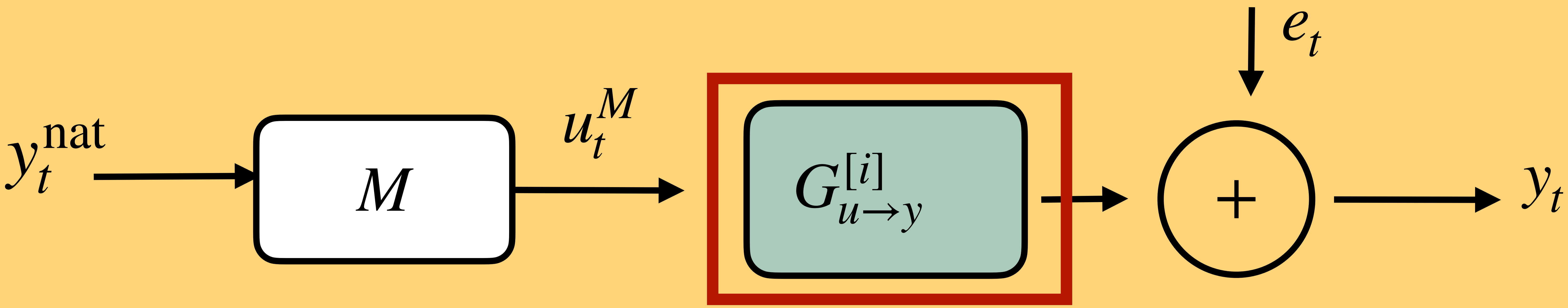

Theorem: $\mathrm{Reg}_T \le \tilde{O}(T^{2/3})$ **where** $T_0 = T^{2/3}$

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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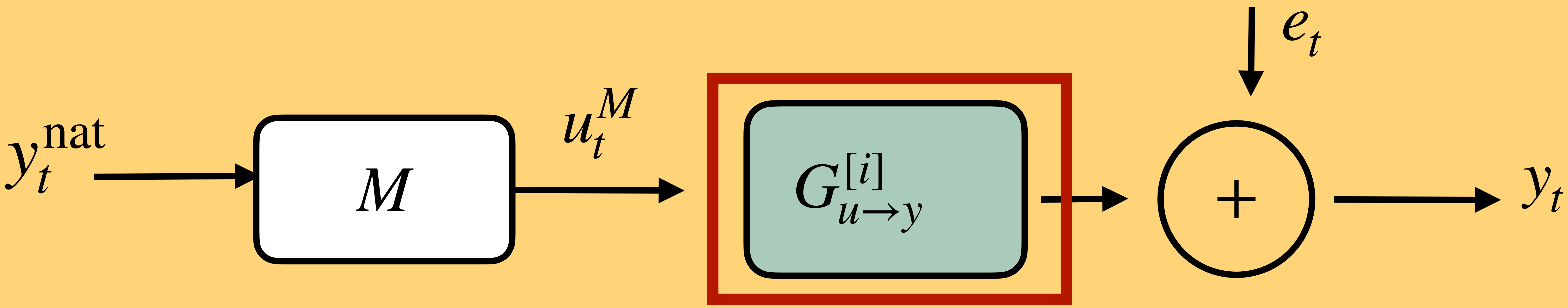

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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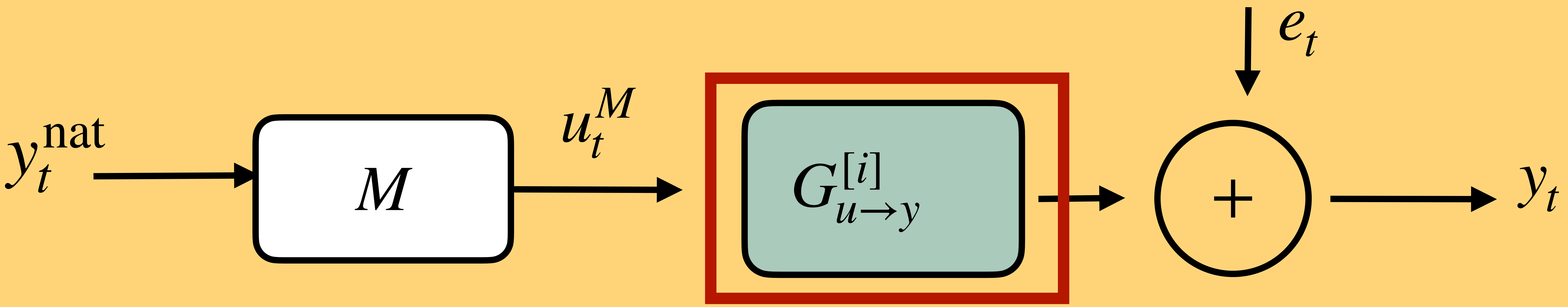

Conveniently: We only ever use and estimate the **Markov operator.**

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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1. We study **unknown dynamics**

2. We combine OCO with **estimating the Markov operator**

3. Everything works just by working with **sequence-to-sequence** , i.e. **improper,** parameterization

Roadmap

4. Optimal Regret: Leveraging Curvature

Fast & Optimal Regret Rates

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Goal: How slow can we make Reg_T as a function of T?

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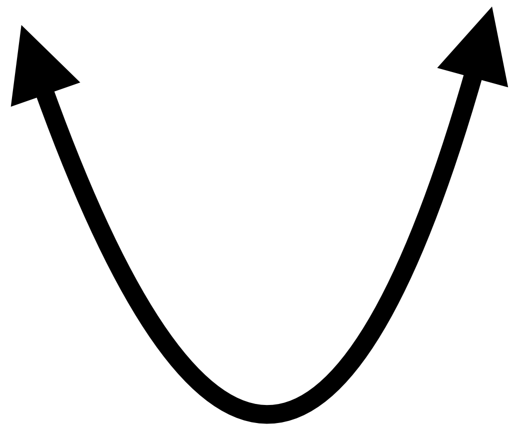

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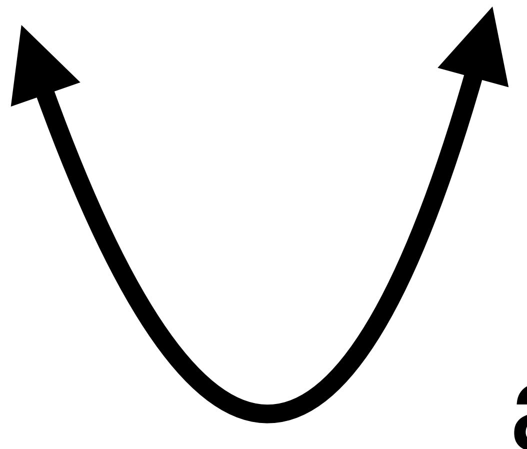

accelerate learning
+ optimization

Fast & Optimal Regret Rates

Theorem: If $c_t(x, u)$ is α-**strongly convex,** there exists algorithms such that

Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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Compare to $\sqrt{T}$ **and** $T^{2/3}$ **regret, previously**

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fixed quadratic cost, i.i.d. Gaussian noise, full observation $y \equiv x_t$

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Takeaway: For s.c. costs, **unknown dynamics determines regret**

changing costs and adversarial noise only affect rates **logarithmically.**

Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

Algorithm: Fast Rates

Optional: Estimate dynamics for first T_0 steps.

For $t = T_0, T_0 + 1, \ldots$

1. $\boxed{u_t \leftarrow u_t^{M_t}}$ **defined in terms of** $M = (M^{[0]}, \ldots, M^{[k]})$

Agrawal, Hazan, Singh "Logarithmic Regret for Online Control", 2019

Simchowitz, Singh, Hazan "Improper Learning for Nonstochastic Control", 2020

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Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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Intuition: Newton solves **ill-conditioned** quadratic functions

Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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Fast rates for unknown dynamics relies on carefully **sensitivity to error argument** + **overparametrization.**

Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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Takeaway: Only thing that changes is the **optimizer + assumptions**

Simchowitz "Making Nonstochastic Control as Easy as Stochastic", 2020

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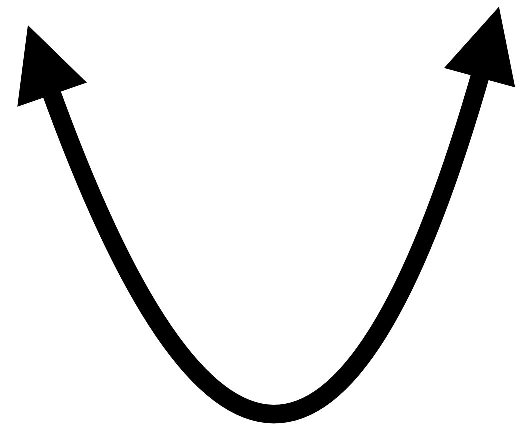
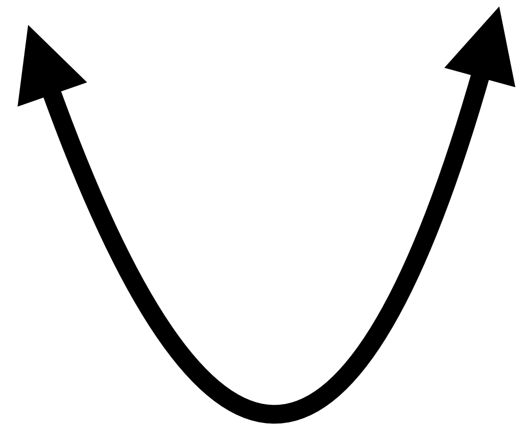


2. With **curvature,** fast rates can be obtained only by **modification of the optimizer.**

Summary



1 **Fast Rates** refer to making Reg_T grow as slow as possible.

2. With **curvature,** fast rates can be obtained only by **modification of the optimizer.**

3. With **curvature,** the regret is determined only by **knowledge of dynamics**, and only logarithmically affected by changing costs + adversarial noise

Hardness Results and Open Questions

Roadmap

5. Open Problems / Hardness Results

The need for stabilization

- **Throughout,** we assumed a **known, stabilizing controller.**

Theorem (Chen & Hazan, ’20)**:** Without a known stabilizing controller, regret is $\Omega(\exp(\text{dimension}))$, until one stabilizes system

Open Question: What are stronger assumptions under one can stabilize the dynamics via online methods?

Beyond linear dynamics

- **Throughout,** we assumed a **fixed, linear dynamics**

Theorem (Gradu, Minyasan, Hazan, ’20)**:** If dynamics A_t, B_t, C_t change **independently** of the learner, then can obtain low **adaptive regret**

Open Question: What if dynamics change **in response to learner?**

Beyond linear dynamics

- **Throughout,** we assumed a **fixed, linear dynamics**

Theorem (Minyasan, Gradu, Simchowitz, Hazan, ’21)**:** If dynamics A_t, B_t, C_t change **independently** of the learner, then can obtain low **adaptive regret**

Open Question: How to learn for truly **nonlinear dynamics?**

Towards practical deployment

- **Thus far,** we have given mostly theoretical results

Theorems: Many of them, illustrating powerful principles in control + AI **(improperness, online learning, adaptation).**

Open Question: Using online control for the **last mile** performance.

Summary

Core Concepts:

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1. From optimal/robust control to regret
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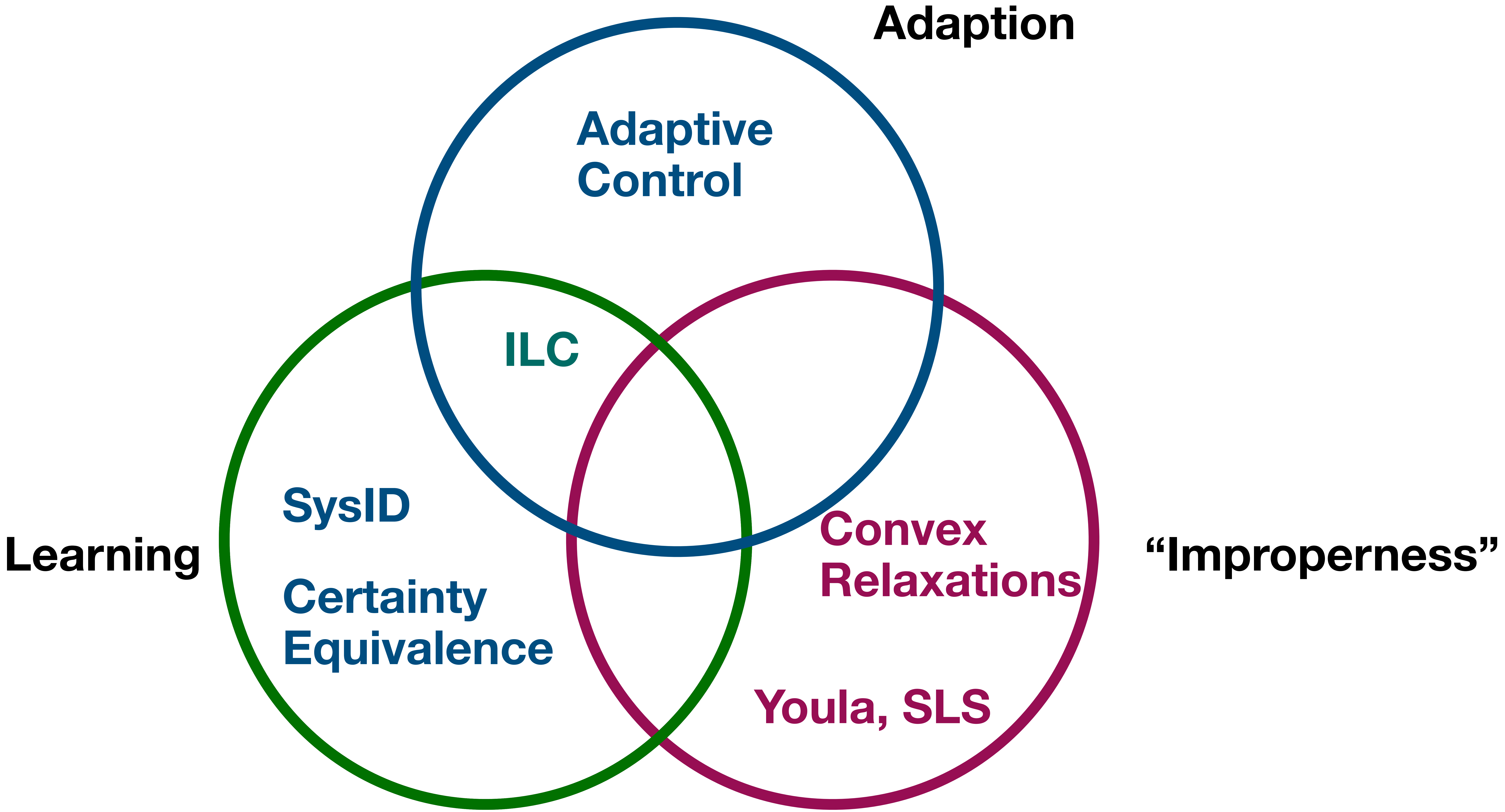
Many open questions!

Non-stochastic control at the intersection



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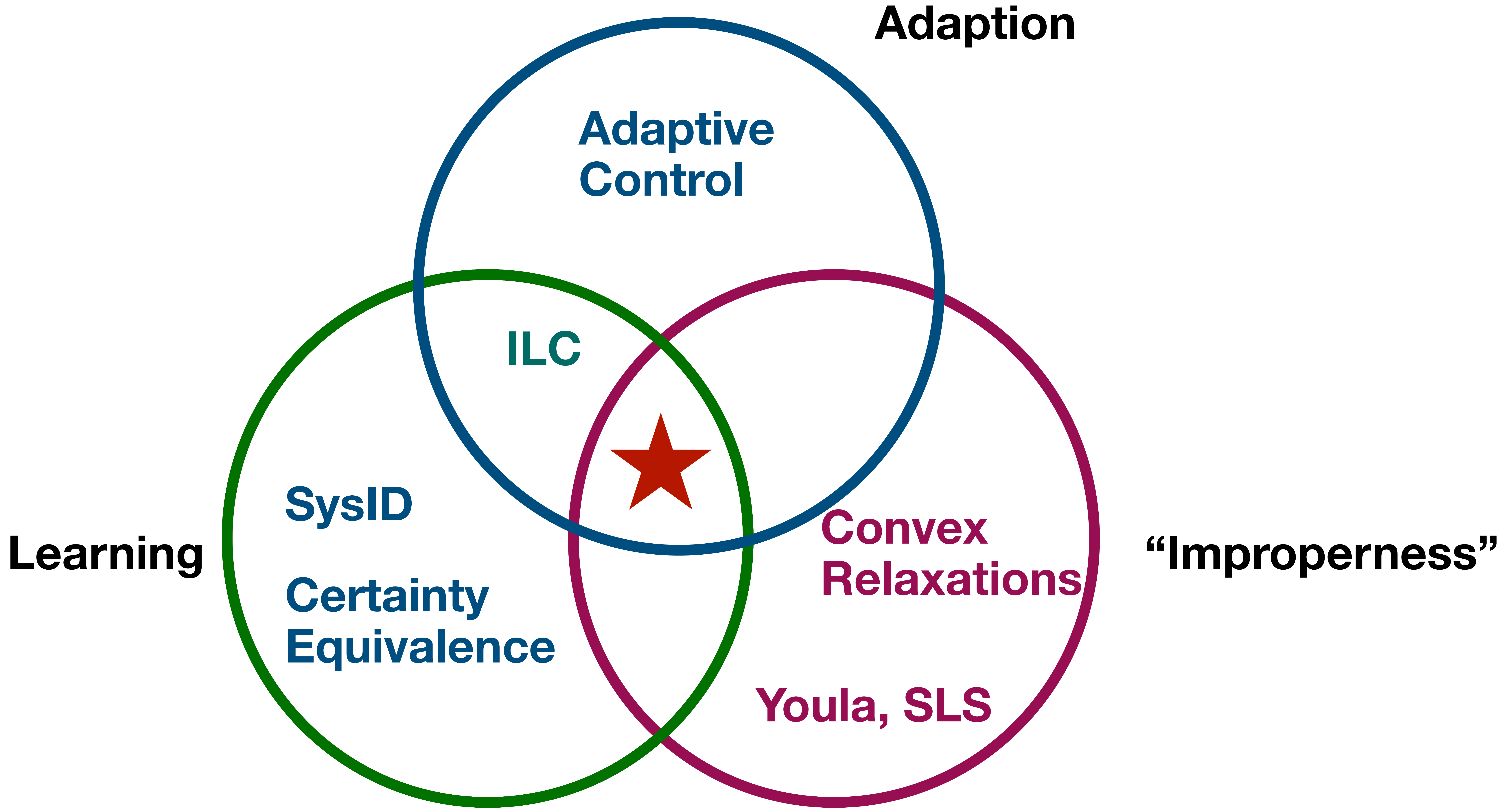

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